The Value of Inside and Outside Money

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We study dynamic economies in which agents may have incentives to hold both privately-issued (*a.k.a.* inside) and publicly-issued (*a.k.a.* outside) circulating liabilities as part of an equilibrium. Our analysis emphasizes spatial separation and limited communication as frictions that motivate monetary exchange. We isolate conditions under which a combination of inside and outside money does and does not allow the economy to achieve a first-best allocation of resources. We also study the extent to which the use of private circulating liabilities alone, or the use of public circulating liabilities alone, can address the frictions that lead to monetary exchange. We identify conditions under which both types of liabilities are essential to efficiency. However, even when these conditions are satisfied, we show that political economy considerations may lead to a prohibition against private circulating liabilities. Finally, we analyze the consequences of such a prohibition for the determinacy of equilibrium, and for endogenously arising volatility.

Key Words: Fiat money, private money, electronic cash, monetary theory, endogenous volatility.

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1. INTRODUCTION

1.1. Overview

Throughout much of monetary history, publicly- and privately-issued circulating liabilities have co-existed. Privately-issued bank notes have circulated alongside specie or greenbacks, and bills of exchange have co-existed with various forms of outside money. And, while it is true that for much of the last seventy years legal restrictions have prevented the private issue of close currency substitutes in the U.S., all legal impediments to private currency issue have recently been repealed. This change in the legal environment has occurred at the same time that it is now technologically feasible to issue a variety of forms of "e-cash," many of which are the electronic equivalent of historically observed private banknotes.[1] Thus we can plausibly expect to see a return to a situation where public and private liabilities circulate side-by-side, and in direct competition with one another.

What should one expect to happen when publicly- and privately-issued currencies or currency substitutes are in common use? The history of monetary theory is replete with competing claims about problems that might emerge—or that might be overcome—when private agents can issue close substitutes for currency. And, indeed, it has been common historically to place a variety of restrictions on private note issue as a means of avoiding problems that such note issue might possibly cause.

Some relatively extreme claims have been made for and against the issue of private circulating liabilities. Hayek (1976), for example, argued that the creation of money should be completely privatized, and that market forces would prevent the over-issue of private notes, the fraudulent issue of notes, and any indeterminacy or volatility of equilibrium that might arise as a result of private note issue.[2] Friedman (1960), on the other hand, asserted that allowing private individuals to issue currency substitutes was a formula for generating indeterminacy of equilibria and "excessive" economic volatility. He argued for legal restrictions that strictly segregate "money" from "credit" markets, so that agents who borrow and lend should not issue circulating liabilities.

[1] For a recent discussion of these issues, see Schreft (1997).

[2] Hayek was also concerned, of course, about government incentives to resort to inflation.

Between these views lies the real bills doctrine. Proponents of that doctrine took the view that the private issue of default-free circulating liabilities poses no clear threat to economic well-being. In particular, such issue does not threaten the determinacy of equilibrium, does not create additional sources of volatility, and does not put upward pressure on the price level. But even so, many ardent advocates of the real bills doctrine—including Adam Smith (1776)—did propose a variety of legal restrictions, including large minimum denomination restrictions, on private note issue.

Somewhat surprisingly, there are few modern theoretical treatments of the coexistence of public and private circulating liabilities. While there is a substantial literature on private monies, little of this literature considers situations where the government and private individuals simultaneously issue circulating liabilities.[3] This situation is important to rectify, because a number of important issues arise when inside and outside money may coexist. For example, if the government is issuing outside money, can there be efficiency gains from allowing private agents to issue currency substitutes? Or, if there are default-free private circulating liabilities, are there potentially efficiency gains from having the government issue outside money? Is the presence of private currency substitutes conducive to the existence of indeterminacies? Does it promote volatility? How does it affect price level determination? We propose to consider these questions.

What features should a model designed to address these issues possess? First, in keeping with a tradition that dates back to Adam Smith (1776), monetary exchange is most interesting in environments where trade is not too centralized. Thus, following Townsend (1980, 1987), we consider an economy in which spatial separation and limited communication forces trade to be undertaken in a variety of separate and distinct markets. In addition, the issue of private circulating liabilities has typically been associated with credit extension. We therefore consider an environment in which agents can borrow and lend. Finally, we consider a situation in which agents' patterns of movement imply that some agents might wish to "borrow" from individuals who they will never meet again. In our envi-

[3]Townsend and Wallace (1987), Williamson (1992, 1999), Champ, Smith, and Williamson (1996), Cavalcanti and Wallace (1999), Smith and Weber (1999), Burdett, Trejos, and Wright (2000), Temzelides and Williamson (2000), and Azariadis, Bullard, and Smith (2000) consider private note issue. Of these papers, only Champ, Smith, and Williamson (1996), Smith and Weber (1999), Williamson (1999), and Azariadis, Bullard, and Smith (2000) consider both inside and outside money.

ronment, such transactions are possible through the use of privately-issued circulating liabilities. In particular, an agent can "borrow" today by issuing a liability to an agent whom he will not meet again. The lender takes the liability and, at some point in the future, trades it to a new agent who will—even further in the future—meet the original issuer of the liability. At that point the liability can be redeemed. Thus a failure of a potential borrower to meet a potential lender again does not preclude the transfer of credit.[4] However, we also allow for patterns of movement that permit some agents to "stay together" over time. For those agents, credit transactions are possible even if there is a prohibition against the use of private circulating liabilities.

We then use this environment to address the following questions. (1) Under what circumstances can a combination of publicly and privately-issued circulating liabilities overcome the frictions associated with spatial separation and limited communication—at least with reference to steady states—and under what circumstances is this not possible? (2) Is there a potential efficiency loss from eliminating outside money, even if a default-free inside money is present? (3) Conversely, is there a potential efficiency loss from prohibiting the use of private circulating liabilities? And, if there is, why have legal restrictions on the issue of private circulating liabilities been so common, historically? (4) Does the use of private circulating liabilities contribute to the indeterminacy of equilibrium? Does it lead to economic volatility that might not otherwise be present? (5) Does the presence of spatial separation and limited communication necessarily enhance the economic role of outside money, as a casual reading of Townsend (1980, 1987) might suggest? (6) Relatedly, when it is necessary to use a combination of public and private liabilities, what does this imply about price level determination?

1.2. Main findings

With respect to the first question, how well a combination of inside and outside money can address the problems created by spatial separation and limited communication depends very much on how the population moves between locations over time. We describe a certain "symmetric itineraries" condition on agents' patterns of relocation under which the simultaneous

[4] Notice the contrast between this situation and that in Kiyotaki and Wright (1989, 1992), where a failure of agents to meet again limits the kinds of trades that can occur.

use of inside and outside circulating liabilities completely overcomes any frictions implied by spatial separation and limited communication. Under this condition, the use of public and private circulating liabilities is a powerful mechanism for achieving a first best allocation of resources. However, when the symmetric itineraries condition is violated, it is generically impossible for any combination of inside and outside money alone to completely undo the consequences of spatial separation and limited communication.[5]

When a combination of inside and outside money does allow the attainment of a first best allocation of resources in our economy, there are potential efficiency losses that might arise from eliminating outside money. These are the conventional efficiency losses that occur in economies of overlapping generations when a non-monetary economy is dynamically inefficient. Thus the analysis is not generally supportive of Hayek's position that money creation is always best left to the market. However, whether *private* circulating liabilities are essential to the attainment of efficiency depends very much on agents' patterns of relocation. In particular, we describe conditions under which a prohibition of private circulating liabilities is completely irrelevant; that is, it has *no* effect on the set of equilibrium prices, returns, or allocations. But, we also describe conditions under which inefficiencies do arise from a prohibition against private circulating liabilities, as well as conditions under which such a prohibition can completely undo the existence of any monetary steady state. Thus, under the appropriate circumstances, a prohibition of private circulating claims can have severe negative consequences for our economy.

Nonetheless, restrictions on the use of private circulating liabilities—or outright prohibitions against them—have been commonly observed throughout history. Why should this be the case? Interestingly, we are able to show that, when a monetary steady state does exist under a prohibition against private currency substitutes, political economy considerations may lead to such a prohibition. This is true even when the use of private circulating liabilities is essential to the attainment of an efficient allocation of resources. In particular, a majority of agents may prefer to undertake a prohibition

[5]In Azariadis, Bullard, and Smith (2000), the use of inside and outside money does generally allow the attainment of a first best allocation of resources, at least with respect to steady states. But their economy implicitly satisfies the type of symmetry condition described in the text.

of private circulating liabilities, even though the agents who benefit from such a prohibition cannot compensate the agents who are harmed by it.

Our results also indicate that the spatial separation and limited communication features of our economy that induce the utilization of private circulating liabilities are a source of considerable volatility that arises from the endogenous dynamics of the economy. Nonetheless, this volatility cannot generally be eliminated by a prohibition against private circulating liabilities, in contrast to the position taken by Friedman (1960). And we find that the use of such liabilities does not create any particular additional sources of indeterminacy. Thus the analysis is, in fact, supportive of the real bills doctrine: Few problems are caused, and some problems may be solved, by allowing private individuals to issue close currency substitutes.

Finally, we show that, in general, the presence of spatial separation and limited communication may *either enhance or reduce* the role for outside money in an economy. More specifically, we demonstrate the existence of economies that have no role for outside money in the absence of spatial separation and limited communication, but that do have a role for outside money when those features are added to the model. Perhaps more surprisingly, we also demonstrate that there are economies that have a role for outside money when spatial separation and limited communication are absent, but that have no such role when those features are present. In short, spatial separation and limited communication can have subtle consequences for the use of outside money. These features of an economy are not guaranteed to promote the use of outside money. As a corollary, we also show that the presence of spatial separation and limited communication—which may lead to the use of private circulating liabilities—can put either upward or downward pressure on an economy's steady state price level.

1.3. Organization

In Section 2 we introduce the environment we consider. We also provide a benchmark for evaluating the consequences of spatial separation and limited communication by describing how our economy would work in the absence of those features. We then provide the conditions that an equilibrium must satisfy in their presence. Also in Section 2 we show how, under a certain symmetric itineraries condition, a combination of inside and outside money can completely overcome the problems of spatial separation and limited communication. In Sections 3, we analyze the consequences of prohibitions against private circulating liabilities when the symmetric itineraries condition is satisfied. In Section 4 we return to the environment studied in Section 2, and consider more general itinerary patterns for the economy. Here the use of public and private liabilities alone cannot completely solve the problems suffered under decentalized exchange. In addition, we show that when symmetry is absent, spatial separation and limited communication can have complicated consequences for the role of outside money. We offer some concluding comments in Section 5.

2. ECONOMIES WITH UNRESTRICTED LIABILITY ISSUE

2.1. The environment

We study a pure exchange economy with overlapping generations of agents who live for three periods on either of two islands. Time is discrete and indexed by $t = 0, 1, 2, ...$ At each date a new young generation appears, consisting of a continuum of agents with unit mass.[6] These agents are divided among islands in a manner described below.

In each period agents consume a single non-storable good. At age k, all agents—regardless of their date or place of birth—receive the endowment $e_k \geq 0$, $k = 1, 2, 3$. Thus each young agent has the lifetime endowment vector (e_1, e_2, e_3). In addition, all agents have the same preferences. Let $c_t(t+s)$, $s = 0, 1, 2$, denote the consumption of an agent born at t in period $t + s$. Then each young agent has the utility function

$$U = \ln c_t(t) + \alpha \ln c_t(t+1) + \beta \ln c_t(t+2) \tag{1}$$

with $\alpha > 0$ and $\beta > 0$.

As in Townsend (1980, 1987), our analysis stresses the consequences of spatial separation and limited communication. More specifically, we assume that at any date resources can only be transferred among agents who inhabit the same location. In addition, as in Townsend (1980, 1987), Townsend and Wallace (1987), and Azariadis, Bullard, and Smith (2000), trade is complicated by the fact that agents move among locations. Let islands be indexed by $1, 2$. Then each young agent is endowed with an itinerary, which we denote by (h, i, j). In particular, $h \in \{1, 2\}$ denotes an agent's place of birth, $i \in \{1, 2\}$ the same agent's location in middle age, and $j \in \{1, 2\}$ denotes the agent's location when old. Following Townsend

[6] Thus, in particular, there is no population growth.

Figure 1

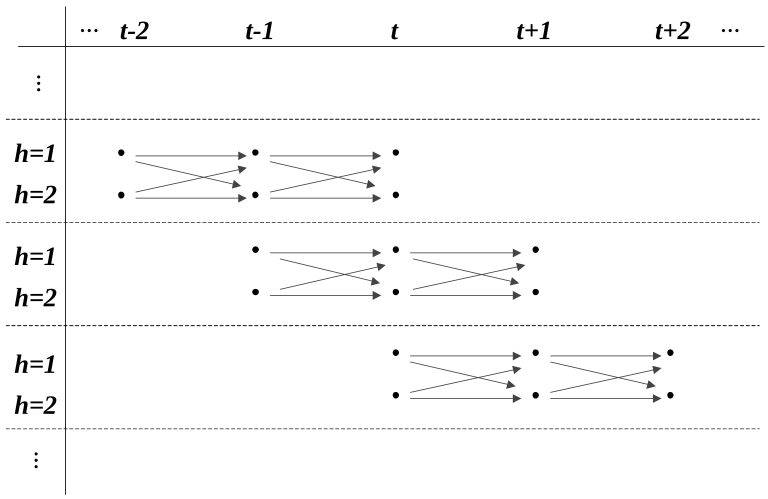

FIG. 1. This schematic diagram shows how generations born at each date may move between the two locations during their lifetimes.

(1980) and Townsend and Wallace (1987), each agent's itinerary is exogenously given. Let $\theta_{hij} > 0$; $h = 1, 2$; $i = 1, 2$; $j = 1, 2$, denote the fraction of each young generation with itinerary (h, i, j). Thus, for example, θ_{111} denotes the fraction of each young generation that remains in location 1 for their entire lives. We require $\sum_{h=1}^{2} \sum_{i=1}^{2} \sum_{j=1}^{2} \theta_{hij} = 1$.

2.2. Savings behavior

As we will see, agents with different itineraries may face different real rates of interest. Thus let an agent born at t with itinerary (h, i, j) face the gross real rate of interest $R_{hi}(t)$ between t and $t+1$, and let $R_{ij}(t+1)$ be the gross real rate of interest faced between $t+1$ and $t+2$. Since $\theta_{hij} > 0$ holds $\forall\ (h, i, j)$ there are always agents with whom any individual can borrow and lend. It follows that agents face the lifetime budget constraint

$$c_t(t) + \frac{c_t(t+1)}{R_{hi}(t)} + \frac{c_t(t+2)}{R_{hi}(t)\,R_{ij}(t+1)} \leq e_1 + \frac{e_2}{R_{hi}(t)} + \frac{e_3}{R_{hi}(t)\,R_{ij}(t+1)}. \tag{2}$$

A young agent maximizing lifetime utility subject to (2) sets

$$c_t(t) = \frac{e_1 + \frac{e_2}{R_{hi}(t)} + \frac{e_3}{R_{hi}(t)R_{ij}(t+1)}}{1+\alpha+\beta} \tag{3}$$

$$\equiv f\left[R_{hi}(t), R_{ij}(t+1)\right],$$

$$c_t(t+1) = \alpha R_{hi}(t) f\left[R_{hi}(t), R_{ij}(t+1)\right], \tag{4}$$

and

$$c_t(t+2) = \beta R_{hi}(t) R_{ij}(t+1) f\left[R_{hi}(t), R_{ij}(t+1)\right]. \tag{5}$$

It will also prove useful to keep track of the asset holdings of young and middle-aged agents at each date. Then, for an agent born at t with itinerary (h, i, j), let $s_1\left[R_{hi}(t), R_{ij}(t+1)\right]$ and $s_2\left[R_{hi}(t), R_{ij}(t+1)\right]$ denote asset holdings between youth and middle age, and between middle and old age, respectively. Clearly,

$$s_1\left[R_{hi}(t), R_{ij}(t+1)\right] = e_1 - f\left[R_{hi}(t), R_{ij}(t+1)\right], \tag{6}$$

and

$$\begin{aligned} s_2\left[R_{hi}(t), R_{ij}(t+1)\right] &\equiv e_2 - \alpha R_{hi}(t) f\left[R_{hi}(t), R_{ij}(t+1)\right] + \\ &\quad R_{hi}(t) s_1\left[R_{hi}(t), R_{ij}(t+1)\right] \\ &\equiv e_1 R_{hi}(t) + e_2 - \\ &\quad (1+\alpha) R_{hi}(t) f\left[R_{hi}(t), R_{ij}(t+1)\right]. \end{aligned} \tag{7}$$

Finally, it will prove useful to keep track of the utility derived by an agent as a function of the rates of interest that he faces during his lifetime. To this end, define the indirect utility function $V\left[R_{hi}(t), R_{ij}(t+1)\right]$ by

$$\begin{aligned} V\left[R_{hi}(t), R_{ij}(t+1)\right] &\equiv \ln\left\{f\left[R_{hi}(t), R_{ij}(t+1)\right]\right\} + \\ &\quad \alpha \ln\left\{\alpha R_{hi}(t) f\left[R_{hi}(t), R_{ij}(t+1)\right]\right\} + \\ &\quad \beta \ln\left\{\beta R_{hi}(t) R_{ij}(t+1) f\left[R_{hi}(t), R_{ij}(t+1)\right]\right\}. \end{aligned} \tag{8}$$

We now establish the following result.

LEMMA 2.1. *Part (a).* $V_1[R_{hi}(t), R_{ij}(t+1)] < (\geq)\ 0$ *holds if* $c_t(t) = f[R_{hi}(t), R_{ij}(t+1)] > (\leq)\ e_1$. *Part (b).* $V_2[R_{hi}(t), R_{ij}(t+1)] > (\leq)\ 0$ *holds if* $c_t(t+2) = \beta R_{hi}(t) R_{ij}(t+1) f[R_{hi}(t), R_{ij}(t+1)] > (\leq)\ e_3$. *Part (c). Define* $\tilde{\mathcal{V}}(R)$ *by* $\tilde{\mathcal{V}}(R) \equiv V(R,R)$. *Then* $\tilde{\mathcal{V}}'(R) > (\leq)\ 0$ *holds if* $0 < (\geq)\ (\alpha+2\beta)e_1 + (\beta-1)\frac{e_2}{R} - (\alpha+2)\frac{e_3}{R^2} \equiv H(R)$.

Proof. See Appendix *A*. ▮

Part (a) of Lemma 2.1 states the intuitive result that utility is decreasing (increasing) in the rate of interest faced when young if a young agent is a borrower (lender). Part (b) states the similarly intuitive result that utility is increasing (decreasing) in the rate of interest faced when middle-aged if the agent is a lender (borrower) at that time. Part (c) of the Lemma states how lifetime utility is affected by changes in the rate of interest faced, if the agent faces the same rate of interest in all periods.

For our purposes, it will be most interesting to consider the situation where agents are net borrowers when young, and net savers when middle aged. Naturally, this requires that

$$s_1[R_{hi}(t), R_{ij}(t+1)] < 0 < s_2[R_{hi}(t), R_{ij}(t+1)], \tag{9}$$

for all (h,i,j), and for all $t \geq 0$.

2.3. The nature of trade

Three types of trades can take place in this economy. First, agents can exchange government-issued (*a.k.a.* outside) fiat money for consumption, as is standard in pure exchange overlapping generations models. We assume that there is a constant outstanding stock of fiat money (per capita) of $\$M$. The distribution of this money across islands can vary over time. Let $M_1(t)$ $[M - M_1(t)]$ be the stock of outside money held in location 1 (2) at t. In addition, let $p_i(t);\ i = 1,2$, denote the price level (the dollar price of a unit of consumption) in location i at date t.

Second, young agents with the itinerary (h,i,j) can borrow or lend with middle-aged agents having either the itinerary $(1,h,i)$ or the itinerary $(2,h,i)$. Here agents are extending credit today to other agents who they will meet again tomorrow. Thus these are "normal" credit transactions, and agents can issue liabilities with a maturity of one period in order to accomplish them. In particular, these liabilities do not circulate.

Third, young agents with the itinerary (h, i, j) can trade with middle-aged agents having the itineraries $(1, h, i')$ or $(2, h, i')$, where $i' = 1\,(2)$ if $i = 2\,(1)$.[7] In transactions of this type, agents enter into borrowing or lending arrangements with other agents who they will never meet again. In order for credit to be extended in these circumstances at date t, young agents must issue liabilities with a maturity of *two periods* to middle-aged agents. The latter then carry these liabilities to a location where they will be separated from their issuers. The two-period liabilities held by agents who are now old at $t + 1$ will then be sold to newly middle-aged agents in location i'. And, they can only be sold to middle-aged agents with itineraries $(1, i', j)$ or $(2, i', j)$, as the agents who buy these liabilities must be able to present them to their issuers for redemption at $t + 2$. Thus liabilities with a maturity of two periods necessarily circulate. In fact, they circulate in much the same way as outside money, and we will refer to privately-issued circulating liabilities as *inside money*, or as a private currency substitute.

It is important to note that a middle-aged agent can acquire any one of four types of liabilities: outside money, one-period (non-circulating) private liabilities, newly-issued two-period (circulating) private liabilities, and previously-issued two-period (circulating) private liabilities. Newly-issued circulating liabilities will be sold next period; previously-issued circulating liabilities will be purchased and presented to their issuer for redemption. It is also deserving of emphasis that borrowing and lending can occur between agents who meet only once. This is possible because the middle-aged agent will—in the future—meet new agents who themselves will—even further in the future—meet the issuers of these liabilities. Thus credit transactions can be mediated through the use of private circulating liabilities. We assume throughout that redemption of all privately-issued liabilities is costlessly enforceable.

2.4. The centralized economy

2.4.1. Overview

In order to better understand the consequences of spatial separation, limited communication, and the use of private circulating liabilities, it is

[7] It is logically possible that young agents with the itinerary $(h, 1, j)$ trade with other young agents having the itinerary $(h, 2, j)$. However, since we focus throughout on situations where all young agents want to borrow, no such transactions will be observed in any equilibrium we consider.

convenient to begin with a brief description of how this economy would function if all agents alive at date t were together in the same location. In other words, we consider an analog economy—henceforth called the "centralized" economy—in which spatial separation and limited communication are not factors. Here we consider only the economy with outside money present. The same economy with no outside money present is described by Azariadis, Bullard, and Ohanian (1999).

2.4.2. *Equilibrium conditions*

In a centralized economy, there is no need to index interest rates by an agent's pattern of movement. Or, in other words, all agents face the common gross rate of interest $R(t)$ between t and $t+1$. Moreover, since there is a single (common) goods market, goods market clearing requires that

$$\begin{aligned} e_1 + e_2 + e_3 \;=\;& f\left[R(t), R(t+1)\right] + \\ & \alpha R(t-1) f\left[R(t-1), R(t)\right] + \\ & \beta R(t-2) R(t-1) f\left[R(t-2), R(t-1)\right]. \end{aligned} \tag{10}$$

It is easy to establish that there is a (monetary) steady state with $R(t) = 1$ $\forall\, t$. Outside money has value (in Gale's (1973) terminology, the economy is Samuelson case) iff

$$s_1(1,1) + s_2(1,1) = \frac{(\alpha + 2\beta) e_1 + (\beta - 1) e_2 - (2 + \alpha) e_3}{1 + \alpha + \beta} > 0. \tag{11}$$

The economy is a classical case economy if (11) fails to hold. In addition, Azariadis, Bullard, and Ohanian (1999) show that (10) has one other solution (a nonmonetary steady state) with $R > 0$. The condition $R < 1$ holds iff (10) is satisfied.

2.4.3. *Local dynamics*

In order to analyze local dynamics, it is convenient to define

$$u(t+1) = R(t) \tag{12}$$

$$v(t+1) = u(t) \tag{13}$$

and to write (10) as

$$\begin{aligned} e_1 + e_2 + e_3 \;=\;& f\left[R(t), R(t+1)\right] + \\ & \alpha u(t) f\left[u(t), R(t)\right] + \\ & \beta v(t) u(t) f\left[v(t), u(t)\right]. \end{aligned} \tag{14}$$

Letting (R, u, v) denote a steady state, we then approximate the dynamical system consisting of (12)-(14) by

$$\begin{bmatrix} R(t+1) - R \\ u(t+1) - u \\ v(t+1) - v \end{bmatrix} = \begin{bmatrix} \frac{\partial R(t+1)}{\partial R(t)} & \frac{\partial R(t+1)}{\partial u(t)} & \frac{\partial R(t+1)}{\partial v(t)} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} R(t) - R \\ u(t) - u \\ v(t) - v \end{bmatrix} \tag{15}$$

with all partial derivatives evaluated at the appropriate steady state.

We can now state a result about local dynamics, in a vicinity of the monetary steady state.

Lemma 2.2. *The matrix J has one eigenvalue in the interval* $(-\infty, -1)$, *one eigenvalue in the interval* $(-1, 0)$, *and one eigenvalue in the interval* $(1, \infty)$.

Proof. See Appendix B. ▮

It is easy to verify that this economy has two free initial conditions. Thus the monetary steady state is determinate, and paths approaching it display damped oscillation. Note in particular that endogenous volatility will be observed.[8]

2.5. Equilibrium in the decentralized economy

2.5.1. Arbitrage conditions

We now return to an analysis of equilibria in an economy where spatial separation and limited communication render trade "decentralized." We begin with a statement of the "no-arbitrage" conditions that an equilibrium must satisfy. An agent following the itinerary $(1,1,1)$ has several options. One is that he can issue a sequence of one-period liabilities, earning the

[8] Azariadis, Bullard, and Ohanian (1999) show that there is a unique steady state in the absence of outside money. It too is determinate, and dynamical equilibria approaching the steady state necessarily display damped oscillation.

interest rates $R_{11}(t)$ and $R_{11}(t+1)$. A second is that he can issue a circulating liability that will be carried from location 1 to location 2 between t and $t+1$, and back to location 1 between $t+1$ and $t+2$. This liability must earn the sequence of one-period returns $R_{12}(t)$ and $R_{21}(t+1)$. The opportunity to issue both kinds of liabilities implies that an arbitrage opportunity will exist unless

$$R_{11}(t)\,R_{11}(t+1) = R_{12}(t)\,R_{21}(t+1) \tag{16}$$

for all $t \geq 0$. For those with itinerary $(1,1,2)$ a similar argument implies that

$$R_{11}(t)\,R_{12}(t+1) = R_{12}(t)\,R_{22}(t+1), \tag{17}$$

for all $t \geq 0$. And, for agents with itineraries $(1,2,2)$ and $(1,2,1)$ we must have

$$R_{12}(t)\,R_{22}(t+1) = R_{11}(t)\,R_{12}(t+1), \tag{18}$$

for all $t \geq 0$ and

$$R_{12}(t)\,R_{21}(t+1) = R_{11}(t)\,R_{11}(t+1) \tag{19}$$

for all $t \geq 0$, respectively. We note that equations (18) and (19) are redundant with equations (16) and (17). The analogous set of no-arbitrage conditions for agents born in location 2 are given by

$$R_{22}(t)\,R_{22}(t+1) = R_{21}(t)\,R_{12}(t+1) \tag{20}$$

for all $t \geq 0$ for those with itinerary $(2,2,2)$,

$$R_{22}(t)\,R_{21}(t+1) = R_{21}(t)\,R_{11}(t+1) \tag{21}$$

for all $t \geq 0$ for those with itinerary $(2,2,1)$,

$$R_{21}(t)\,R_{11}(t+1) = R_{22}(t)\,R_{21}(t+1) \tag{22}$$

for all $t \geq 0$ for agents following the itinerary $(2,1,1)$, and finally

$$R_{21}(t)\,R_{12}(t+1) = R_{22}(t)\,R_{22}(t+1) \tag{23}$$

for all $t \geq 0$ for agents with itinerary $(2,1,2)$. Again, equation (22) is redundant with equation (21), while equation (23) is redundant with equation (20).

In addition, since agents who are net savers (middle-aged agents) have the option of carrying outside (fiat) money or claims against private individuals, an absence of arbitrage opportunities requires that inside and outside money bear the same rate of return. Hence

$$\frac{p_i(t)}{p_j(t+1)} = R_{ij}(t) \tag{24}$$

for all $i = 1, 2$; $j = 1, 2$; $t \geq 0$.

We now observe that there is substantial redundancy even among the no arbitrage conditions (16), (17), (20), (21), and (24). A formal result is stated in Lemma 2.3.

LEMMA 2.3. *The independent no-arbitrage conditions are (16), (20), and (24).*

Proof. See Appendix *C*. ▮

2.5.2. Market clearing

Market clearing in location 1 at time t requires that

$$A_1 + B_1 + C_1 = E_1, \tag{25}$$

where A_1 is the desired consumption of the young in location 1 at date t, given by

$$\begin{aligned} A_1 \equiv\ & \theta_{111} f[R_{11}(t), R_{11}(t+1)] + \theta_{112} f[R_{11}(t), R_{12}(t+1)] + \\ & \theta_{121} f[R_{12}(t), R_{21}(t+1)] + \theta_{122} f[R_{12}(t), R_{22}(t+1)], \end{aligned} \tag{26}$$

B_1 is the desired consumption of the middle-aged in location 1 at date t, given by

$$\begin{aligned} B_1 \equiv\ & \theta_{111} \alpha R_{11}(t-1) f[R_{11}(t-1), R_{11}(t)] + \\ & \theta_{112} \alpha R_{11}(t-1) f[R_{11}(t-1), R_{12}(t)] + \\ & \theta_{212} \alpha R_{21}(t-1) f[R_{21}(t-1), R_{12}(t)] + \\ & \theta_{211} \alpha R_{21}(t-1) f[R_{21}(t-1), R_{11}(t)], \end{aligned} \tag{27}$$

C_1 is the desired consumption of the old in location 1 at date t, given by

$$\begin{aligned} C_1 \equiv\ & \theta_{111}\beta R_{11}(t-2) R_{11}(t-1) f[R_{11}(t-2), R_{11}(t-1)] + \\ & \theta_{121}\beta R_{12}(t-2) R_{21}(t-1) f[R_{12}(t-2), R_{21}(t-1)] + \\ & \theta_{211}\beta R_{21}(t-2) R_{11}(t-1) f[R_{21}(t-2), R_{11}(t-1)] + \\ & \theta_{221}\beta R_{22}(t-2) R_{21}(t-1) f[R_{22}(t-2), R_{21}(t-1)], \end{aligned} \tag{28}$$

and E_1 is the available endowment in location 1 at date t, given by

$$\begin{aligned} E_1 \equiv\ & (\theta_{111} + \theta_{112} + \theta_{121} + \theta_{122}) e_1 + \\ & (\theta_{111} + \theta_{112} + \theta_{212} + \theta_{211}) e_2 + \\ & (\theta_{111} + \theta_{121} + \theta_{211} + \theta_{221}) e_3 \end{aligned} \tag{29}$$

Similarly, market clearing in location 2 at time t requires

$$A_2 + B_2 + C_2 = E_2 \tag{30}$$

where A_2 is the desired consumption of the young in location 2 at date t, given by

$$\begin{aligned} A_2 \equiv\ & \theta_{222} f[R_{22}(t), R_{22}(t+1)] + \theta_{221} f[R_{22}(t), R_{21}(t+1)] + \\ & \theta_{212} f[R_{21}(t), R_{12}(t+1)] + \theta_{211} f[R_{21}(t), R_{11}(t+1)], \end{aligned} \tag{31}$$

B_2 is the desired consumption of the middle-aged in location 2 at date t, given by

$$\begin{aligned} B_2 \equiv\ & \theta_{222}\alpha R_{22}(t-1) f[R_{22}(t-1), R_{22}(t)] + \\ & \theta_{221}\alpha R_{22}(t-1) f[R_{22}(t-1), R_{21}(t)] + \\ & \theta_{122}\alpha R_{12}(t-1) f[R_{12}(t-1), R_{22}(t)] + \\ & \theta_{121}\alpha R_{12}(t-1) f[R_{12}(t-1), R_{21}(t)], \end{aligned} \tag{32}$$

C_2 is the desired consumption of the old in location 2 at date t, given by

$$\begin{aligned} C_2 \equiv\ & \theta_{222}\beta R_{22}(t-2)\,R_{22}(t-1)\,f\left[R_{22}(t-2),R_{22}(t-1)\right]+ \\ & \theta_{212}\beta R_{21}(t-2)\,R_{12}(t-1)\,f\left[R_{21}(t-2),R_{12}(t-1)\right]+ \\ & \theta_{112}\beta R_{11}(t-2)\,R_{12}(t-1)\,f\left[R_{11}(t-2),R_{12}(t-1)\right]+ \\ & \theta_{122}\beta R_{12}(t-2)\,R_{22}(t-1)\,f\left[R_{12}(t-2),R_{22}(t-1)\right], \end{aligned} \tag{33}$$

and E_2 is the available endowment in location 2 at date t, given by

$$\begin{aligned} E_2 \equiv\ & (\theta_{222}+\theta_{221}+\theta_{212}+\theta_{211})\,e_1+ \\ & (\theta_{222}+\theta_{221}+\theta_{122}+\theta_{121})\,e_2+ \\ & (\theta_{222}+\theta_{212}+\theta_{122}+\theta_{112})\,e_3. \end{aligned} \tag{34}$$

The "no-arbitrage" conditions (16) and (20), along with the two goods market clearing conditions (25) and (30), constitute the equilibrium laws of motion for the interest rate sequence $\{R_{11}(t),R_{12}(t),R_{21}(t),R_{22}(t)\}$. Note that since equations (25) and (30) are third-order difference equations, the equilibrium conditions of our economy potentially constitute a 12^{th}-order dynamical system.

Our intention is to provide a relatively complete analysis of steady state equilibria and of local dynamics (in a neighborhood of any steady state) in full generality. However, a certain "symmetry" assumption on the fraction of the population following certain itineraries will substantially simplify the analysis. Hence we begin with a consideration of this case. We analyze steady state equilibria and and local dynamics near steady states when this symmetry assumption is not satisfied later in the paper.

2.5.3. Symmetry in itineraries

Recall that, for agents who are born in location 1, there are four possible itineraries: $(1,1,1)$, $(1,1,2)$, $(1,2,1)$, and $(1,2,2)$. We denote the fraction of the population following each of these itineraries by θ_{111}, θ_{112}, θ_{121}, and θ_{122} respectively. We impose no conditions on these values other than that they are non-negative. Similarly, for agents born in location 2 there are four possible itineraries: $(2,2,2)$, $(2,2,1)$, $(2,1,2)$, and $(2,1,1)$. We let θ_{222}, θ_{221}, θ_{212}, and θ_{211} denote the fraction of the population following each of these itineraries. Again, to this point no assumptions have been imposed on these itineraries other than non-negativity and $\sum_h \sum_i \sum_j \theta_{hij} = 1$. We now impose a symmetry condition: specifically, that $\theta_{111} = \theta_{222}$, $\theta_{121} = \theta_{212}$, $\theta_{122} = \theta_{211}$, and $\theta_{112} = \theta_{221}$ hold. Thus, in particular, the fraction of agents following any pattern of movement originating in location 1 is equal to the fraction of agents following the same pattern of movement originating in location 2. Note also that symmetry implies $\theta_{111} + \theta_{112} + \theta_{121} + \theta_{122} = \frac{1}{2}$ and similarly that $\theta_{222} + \theta_{221} + \theta_{212} + \theta_{211} = \frac{1}{2}$. We now analyze steady state equilibria, and local dynamics in a neighborhood of a steady state, under this *symmetric itineraries* assumption.

2.5.4. Steady state equilibria

When $R_{ij}(t) = R_{ij}\ \forall\ t$, the no-arbitrage conditions (16) and (20) imply that $R_{11} = R_{22} = R$, and that $R_{12}R_{21} = R^2$ must hold. In addition, when our symmetry assumption is satisfied, we have the following result.

Proposition 2.1. *Under the symmetric itineraries assumption, there generically exists a steady state with* $R_{12} = R_{21} = R = 1$, *and a steady state with* $R_{12} = R_{21} = R \neq 1$. *The equilibrium value of* R *in the latter steady state coincides with that obtaining in the non-monetary steady state of the centralized economy. Thus* $R < (>)\ 1$ *holds in a Samuelson (classical) case economy.*

Proof. See Appendix D. ∎

Proposition 2.1 asserts that, in an economy whose centralized analog is Samuelsonian, there exists a steady state where the combination of public and private circulating liabilities completely overcomes the frictions implied by spatial separation and limited communication. That is, there is a steady state where the decentralized economy attains the same, Pareto optimal, allocation as its centralized analog. And, in addition, private circulating liabilities alone can, in the steady state with $R \neq 1$, permit the attainment of the allocation that prevails in the non-monetary steady state of the centralized analog economy. That steady state is Pareto optimal if $R > 1$. Thus the use of circulating liabilities can permit the economy to deal very effectively with the problem of spatial separation and limited communication.[9]

[9] As we will see this is not typically the case if our symmetry assumption is violated.

As is conventional in the overlapping generations literature, we will call the steady state with $R = (\neq)\ 1$ a monetary (non-monetary) steady state. In a monetary steady state, the price levels in each location are determined as follows. First, since the money supply is unchanging over time, $p_1(t) = p_1$ and $p_2(t) = p_2$ are satisfied. Second, the no-arbitrage condition (24) implies that $R_{12} = p_1/p_2 = 1$, so that $p_1 = p_2$ necessarily holds. Not surprisingly, given symmetry, each location has the same price level. Finally, the net demand for assets in location 1 must equal the value of the outside money circulating in location 1, and similarly in location 2. This net demand is location 1 is

$$\theta_{111}s_2(R,R) + \theta_{112}s_2(R,R_{12}) + \theta_{211}s_2(R_{21},R) + \theta_{212}s_2(R_{21},R_{12}) +$$

$$\theta_{111}s_1(R,R) + \theta_{112}s_1(R,R_{12}) + \theta_{121}s_1(R_{12},R_{21}) + \theta_{122}s_1(R_{12},R)$$

$$= \left(\frac{1}{2}\right)[s_2(1,1) + s_1(1,1)] \tag{35}$$

by symmetry and Proposition 2.1. Moreover,

$$s_2(1,1) + s_1(1,1) = [e_1 + e_2 - (1+\alpha) f(1,1)] + e_1 - f(1,1) \tag{36}$$

$$= \frac{(\alpha + 2\beta) e_1 + (\beta - 1) e_2 - (2+\alpha) e_3}{1+\alpha+\beta}.$$

Thus

$$\frac{M_1(t)}{p_1} = \left(\frac{1}{2}\right)\left\{\frac{(\alpha + 2\beta) e_1 + (\beta - 1) e_2 - (2+\alpha) e_3}{1+\alpha+\beta}\right\}. \tag{37}$$

By symmetry, the net demand for assets in location 2 is equal to that in location 1, in a monetary steady state. Thus, in location 2 we must have

$$\frac{M_2(t)}{p_2} = \frac{M - M_1(t)}{p_1} = \left(\frac{1}{2}\right)\left\{\frac{(\alpha + 2\beta) e_1 + (\beta - 1) e_2 - (2+\alpha) e_3}{1+\alpha+\beta}\right\}. \tag{38}$$

Adding equations (37) and (38) gives

$$\frac{M}{p_1} = \frac{(\alpha + 2\beta) e_1 + (\beta - 1) e_2 - (2+\alpha) e_3}{1+\alpha+\beta}. \tag{39}$$

We therefore have the following result.

Proposition 2.2. *There is a monetary steady state in the decentralized economy (that is, $p_1 > 0$ holds) iff its centralized analog is a Samuelson case economy.*

Proof. See Appendix E. ▮

As we will see, Proposition 2.2 is also not generally true if our symmetry assumption is violated.

Finally, while we have no formal results on the number of steady state equilibria, we report that an extensive numerical analysis failed to reveal any steady states other than those described in Proposition 2.1. We therefore conjecture that, when our symmetry assumption holds, there are no additional steady state equilibria. If this is correct, $R_{12} = R_{21} = R$ must hold in any steady state, and all agents face the same rates of return, regardless of itineraries. Once again, this will not generally be the case if the symmetry assumption fails.

2.5.5. Local dynamics

The dimensionality of the dynamical system consisting of (16), (20), (25) and (30) is too large for us to derive any analytical results regarding dynamics even in a neighborhood of either of the steady states described in Proposition 2.1. However, we did linearize that system and compute eigenvalues of the appropriate Jacobian matrix numerically, at both the monetary and the non-monetary steady state.[10] Our calculations were conducted as follows. Under our symmetric itineraries assumption, we have nine parameters in our system—the endowment triple, the preference parameters α and β, and four itinerary masses. We chose values for these parameters randomly, in order to create a set of 1,000 economies, each one parameterized differently. For the endowment pattern, we began by choosing values r_1, r_2, and r_3 according to draws from a uniform distribution on $[0, 1]$. We then set $e_i = r_i/(r_1 + r_2 + r_3)$. This procedure allows us to "span the space" of possible relative endowments. We chose the itinerary masses in a similar fashion, choosing ℓ_1, ℓ_2, ℓ_3, and ℓ_4 as draws from a uniform distribution defined on $[0, 1]$, and then setting, for instance, $\theta_{111} = \theta_{222} = \ell_1/(2 * (\ell_1 + \ell_2 + \ell_3 + \ell_4))$, and so on. We set α and β directly according to draws from a uniform distribution on $[0, 1]$. For each of the $1,000$ economies we created, we searched for steady states. (We

[10] The local dynamics near the non-monetary steady state are "hyperinflationary dynamics" where outside money is asymptotically becoming valueless.

always found exactly two steady states, which correspond to the monetary and nonmonetary steady states of our analysis.) We also linearized the dynamic system and evaluated the resulting Jacobian matrix at each steady state. The eigenvalues of that matrix then contain information concerning the local dynamics of the system in a neighborhood of the steady state.

The results are as follows. First, if the centralized analog economy is Samuelsonian, then at the monetary steady state two eigenvalues are zero, and five other eigenvalues lie inside the unit circle. Complex and/or negative eigenvalues can be observed. In addition, -1 is always an eigenvalue.[11] Finally, four eigenvalues lie outside the unit circle.

It is straightforward to establish that this economy has eight given initial conditions. Since the steady state is not hyperbolic, our calculation of eigenvalues does not enable us to ascertain whether the monetary steady state can be approached. To do so would require us to numerically approximate the center manifold.[12] The dimensionality of our system is too large for this to be feasible. However, in a simpler but related economy, Azariadis, Bullard, and Smith (2000) do numerically approximate the center manifold, and in Samuelson case economies they establish that dynamics are stable along it. Based on their result, we conjecture that dynamics along the center manifold are stable in a neighborhood of the monetary steady state here as well. If that conjecture is correct, then the monetary steady state can be approached, and the monetary steady state is determinate. Thus there will be a unique dynamical equilibrium path that approaches the monetary steady state. Note that oscillation will generically be observed along such paths. While this is also true of paths approaching the monetary steady state in the centralized analog economy, here the presence of an eigenvalue equal to -1 implies that oscillation will generally dampen only extremely slowly.

With respect to the non-monetary steady state in a Samuelson case economy, the Jacobian matrix has two zero eigenvalues, and four eigenvalues in the interior of the unit circle. Again, -1 is an eigenvalue,[13] and there are four eigenvalues outside the unit circle. Thus the non-monetary steady state can be approached. Moreover, if the dynamics are stable along the center manifold, then there is a one-dimensional indeterminacy of equilib-

[11] In a simpler, but conceptually similar economy with spatial separation and limited communication, Azariadis, Bullard, and Smith (2000) establish analytically that -1 is always an eigenvalue.

[12] That is, the manifold associated with the eigenvalue -1. See Wiggins (1990) for a discussion of how to numerically approximate a center manifold.

[13] This is also true in Azariadis, Bullard, and Smith (2000).

ria approaching the non-monetary steady state. As this is the situation found by Azariadis, Bullard, and Smith (2000), we conjecture that this is the case. And, obviously, dynamical equilibrium paths approaching the non-monetary steady state will generically display endogenous oscillation.

For classical case economies, the Jacobian matrix evaluated at the monetary steady state has two zero eigenvalues, and six eigenvalues inside the unit circle. These may include negative and/or complex eigenvalues. Once again, -1 is an eigenvalue, and three eigenvalues lie outside the unit circle. We can conclude that dynamical equilibrium paths approaching the monetary steady state exist, and that such paths will generically display oscillation.

Finally, at the nonmonetary steady state of classical case economies, the Jacobian matrix has two zero eigenvalues, and five eigenvalues inside the unit circle. An eigenvalue equal to -1 remains present, and there are four eigenvalues outside the unit circle. Azariadis, Bullard, and Smith (2000) did not consider an analog to this case, but we conjecture that the dynamics are stable along the center manifold. If this conjecture is correct, the nonmonetary steady state is determinate in this case. Equilibrium paths approaching it display endogenously arising volatility.

2.5.6. The role of private circulating liabilities

What role do private circulating liabilities play in this economy? How essential are they in overcoming the frictions implied by spatial separation and limited communication? The answers to these questions turn out to depend critically on the values of θ_{hij} for $h \in \{1,2\}$; $i \in \{1,2\}$; and $j \in \{1,2\}$; even under our symmetry assumption. We now consider several possibilities in this regard.

As we have already noted, in order for outside money to be valued in each location, the savings of middle-aged borrowers in that location must exceed the credit demand of young agents in the same location. Under our symmetry condition this requires that the centralized analog economy be Samuelsonian.

Now suppose that the following condition holds:

$$\theta_{111}\left[e_1 + e_2 - (1+\alpha) f(1,1)\right] +$$

$$\theta_{211}\left[e_1 R_{21} + e_2 - (1+\alpha) R_{21} f(R_{21},1)\right] +$$

$$\theta_{111}\left[e_1 - f(1,1)\right] + \theta_{112}\left[e_1 - f(1,R_{12})\right] = \tag{40}$$

$$(\theta_{111}+\theta_{211})\left[e_1+e_2-(1+\alpha)\,f\,(1,1)\right]+$$

$$(\theta_{111}+\theta_{112})\left[e_1-f\,(1,1)\right]\geq 0$$

where we have used the fact that, under our symmetric itineraries assumption, $R_{12}=1$ in a monetary steady state. Equation (40) asserts that the savings of middle-aged agents staying in location 1 is adequate to satisfy the credit demand of young agents staying in location 1. We will refer to this as "the market $1\mapsto 1$." If equation (40) holds, young agents with itineraries $(1,1,j)$ do not need to issue circulating liabilities; they can simply issue one period liabilities to agents they will meet again in the next period. If

$$\begin{aligned}&\theta_{112}\left[e_1+e_2-(1+\alpha)\,f\,(1,R_{12})\right]+\\&\theta_{212}\left[e_1R_{21}+e_2-(1+\alpha)\,R_{21}f\,(R_{21},R_{12})\right]+\\&\theta_{121}\left[e_1-f\,(R_{12},R_{21})\right]+\theta_{122}\left[e_1-f\,(R_{12},1)\right]=\\&(\theta_{112}+\theta_{212})\left[e_1+e_2-(1+\alpha)\,f\,(1,1)\right]+\\&(\theta_{121}+\theta_{122})\left[e_1-f\,(1,1)\right]\geq 0\end{aligned}\tag{41}$$

also holds, the same is true for "the market $1\mapsto 2$." Hence, if (40) and (41) both hold, no agent in location 1 needs to issue circulating liabilities. We note that, if the centralized analog economy is Samuelsonian, then at least one of the conditions (40) or (41) must hold as a strict inequality. We also note that if (40) [(41)] *fails* to hold, the credit demand of young agents in "the market $1\mapsto 1$" ("the market $1\mapsto 2$") exceeds the saving of middle-aged agents in the same market. Hence young agents in "the market $1\mapsto 1$" ("the market $1\mapsto 2$") must issue some liabilities that are held by agents they will *not* meet next period. That is, young agents in the appropriate market must issue private circulating liabilities.

The analogous conditions to (40) and (41) in location 2 are

$$\begin{aligned}&\theta_{222}\left[e_1+e_2-(1+\alpha)\,f\,(1,1)\right]+\\&\theta_{122}\left[e_1R_{12}+e_2-(1+\alpha)\,R_{12}f\,(R_{12},1)\right]+\\&\theta_{222}\left[e_1-f\,(1,1)\right]+\theta_{221}\left[e_1-f\,(1,R_{21})\right]=\\&(\theta_{222}+\theta_{122})\left[e_1+e_2-(1+\alpha)\,f\,(1,1)\right]+\\&(\theta_{222}+\theta_{221})\left[e_1-f\,(1,1)\right]\geq 0\end{aligned}\tag{42}$$

and

$$\begin{aligned}
&\theta_{221}\left[e_1+e_2-(1+\alpha) f(1, R_{21})\right]+ \\
&\theta_{121}\left[e_1 R_{12}+e_2-(1+\alpha) R_{12} f(R_{12}, R_{21})\right]+ \\
&\theta_{212}\left[e_1-f(R_{21}, R_{12})\right]+\theta_{211}\left[e_1-f(R_{21}, 1)\right]= \\
&(\theta_{221}+\theta_{121})\left[e_1+e_2-(1+\alpha) f(1,1)\right]+ \\
&(\theta_{212}+\theta_{211})\left[e_1-f(1,1)\right] \geq 0.
\end{aligned} \tag{43}$$

Again, if the centralized analog economy is Samuelsonian, at least one of the inequalities (42) and (43) must hold strictly.

Based on this analysis, we deduce that the nature of the monetary steady state in our economy can be any of several types.

- Case 1: Inequalities (40), (41), (42) and (43) all hold. In this case no agents need to issue circulating liabilities.
- Case 2: Inequalities (40) and (42) hold while inequalities (41) and (43) are violated. The issuers of circulating liabilities are young agents who will change location.
- Case 3: Inequalities (41) and (43) hold while inequalities (40) and (42) are violated. The issuers of circulating liabilities are young agents who will stay in the same location.

Cases 1, 2, and 3 can occur for any itinerary pattern, even itineraries satisfying our symmetric itineraries assumption. If the symmetry assumption is relaxed, one could also find situations where outside money is valued in each location, while:

- Case 4: Young agents who will move from location 1 to location 2 issue circulating liabilities, while young agents born in location 2 who will remain there also issue circulating liabilities.
- Case 5: Only agents who move from location 1 to location 2, or conversely, issue circulating liabilities.

Notice that, in case 1, private circulating liabilities play no role whatsoever. Savings in each "market $i \mapsto j$" is adequate to meet credit demand in that market. No agents need to issue liabilities that will change hands. However, if there is any market where there is an excess demand for credit, privately-issued circulating liabilities are needed to address the problems of spatial separation and limited communication.

3. PROHIBITION OF PRIVATE CIRCULATING LIABILITIES WITH OUTSIDE MONEY VALUED

3.1. Overview

As noted in the introduction, it has often been argued that private agents should be prohibited from issuing liabilities that circulate in a way that competes with government-issued currency. Two particular concerns are that a failure to segregate "money" from "credit" markets can lead to indeterminacies (multiplicity of equilibria) and "excessive" volatility.[14] We now analyze the behavior of an economy with spatial separation and limited communication when private agents are precluded from issuing liabilities that circulate. Throughout we focus our attention on economies where there is a positive (and constant) stock of outside money that has value in equilibrium. We restrict the analysis to economies that satisfy our symmetric itineraries assumption. We offer a few remarks at the end of the section concerning economies that violate this assumption.

3.2. A Case 1 economy

In a Case 1 economy, conditions (40), (41), (42) and (43) all hold. As a consequence, no agent needs to issue circulating liabilities, even if their issue is permitted. Consequently, a prohibition against private circulating liabilities is *irrelevant*. Such a prohibition does not affect the set of equilibria, equilibrium rates of return, or equilibrium price level paths. Thus a prohibition against private circulating liabilities has no costs in this case, and, in particular, such a prohibition does not adversely affect welfare. At the same time such a prohibition has no obvious benefits. It improves the welfare of no agent, and it has no benefits from the perspectives of the determinacy of equilibrium or of reducing the potential for endogenous volatility. Finally, a prohibition on private circulating liabilities does not even affect the price level. Private currency issue, in a Case 1 economy, is not a factor that tends to raise prices.

3.3. A Case 2 economy

3.3.1. Equilibrium conditions

In a Case 2 economy, conditions (40) and (42) hold while conditions (41) and (43) are violated. Thus, if the issue of private circulating liabilities is permitted, these liabilities need only be issued by young agents in "the market $1 \mapsto 2$" and "the market $2 \mapsto 1$." Therefore, under a prohibition against the issue of private circulating liabilities, it is natural to conjecture

[14] See Sargent and Wallace (1982) for a modern discussion of these issues.

the existence of an equilibrium where all the savings of middle-aged agents in "the market $1 \mapsto 2$" and "the market $2 \mapsto 1$" are absorbed by purchasing one-period liabilities from young agents in the same market.[15] In such an equilibrium, all outside money must then be held by middle-aged agents in "the market $1 \mapsto 1$" and "the market $2 \mapsto 2$." It follows that the stock of outside money is unchanging over time in each location. Or, in other words, no outside money is ever carried from one location to another by any agent.

Let M_1 (M_2) be the initial outside money stock in location 1 (2), respectively, with $M_1 + M_2 = M$. Then $M_i(t) = M_i$, $i = 1, 2$, $\forall\, t \geq 0$. Moreover, in the equilibrium sought, we have the following equilibrium conditions. In "the market $1 \mapsto 2$," the demand for credit by young agents absorbs the entire supply of savings by middle-aged agents. Thus

$$\theta_{112}\left[e_1 R_{11}(t-1) + e_2 - (1+\alpha) R_{11}(t-1) f\left[R_{11}(t-1), R_{12}(t)\right]\right] +$$

$$\theta_{212}\left[e_1 R_{21}(t-1) + e_2 - (1+\alpha) R_{21}(t-1) f\left[R_{21}(t-1), R_{12}(t)\right]\right] +$$

$$\theta_{121}\left[e_1 - f\left[R_{12}(t), R_{21}(t+1)\right]\right] + \theta_{122}\left[e_1 - f\left[R_{12}(t), R_{22}(t+1)\right]\right] = 0. \tag{44}$$

Similarly, in "the market $2 \mapsto 1$,"

$$\theta_{221}\left[e_1 R_{22}(t-1) + e_2 - (1+\alpha) R_{22}(t-1) f\left[R_{22}(t-1), R_{21}(t)\right]\right] +$$

$$\theta_{121}\left[e_1 R_{12}(t-1) + e_2 - (1+\alpha) R_{12}(t-1) f\left[R_{12}(t-1), R_{21}(t)\right]\right] +$$

$$\theta_{212}\left[e_1 - f\left[R_{21}(t), R_{12}(t+1)\right]\right] + \theta_{211}\left[e_1 - f\left[R_{21}(t), R_{11}(t+1)\right]\right] = 0. \tag{45}$$

In "the market $1 \mapsto 1$," on the other hand, the savings of middle-aged agents is more than adequate to meet the demand for loans by young agents. Hence outside money is held by middle-aged agents in this market, and

$$\frac{M_1}{p_1(t)} = \theta_{111}[e_1 R_{11}(t-1) + e_2 -$$

$$(1+\alpha) R_{11}(t-1) f[R_{11}(t-1), R_{11}(t)]] +$$

$$\theta_{211}[e_1 R_{21}(t-1) + e_2 - \tag{46}$$

[15] Recall in particular that these are the markets where there is an excess demand for credit in a Case 2 economy.

$$(1+\alpha) R_{21}(t-1) f[R_{21}(t-1), R_{11}(t)]]+$$

$$\theta_{111}[e_1 - f[R_{11}(t), R_{11}(t+1)]] +$$

$$\theta_{112}[e_1 - f[R_{11}(t), R_{12}(t+1)]],$$

where $p_1(t)$ is the price level in location 1. Finally, for "the market $2 \mapsto 2$" we have the analogous condition

$$\frac{M_2}{p_2(t)} = \theta_{222}[e_1 R_{22}(t-1) + e_2 -$$

$$(1+\alpha) R_{22}(t-1) f[R_{22}(t-1), R_{22}(t)]]+$$

$$\theta_{122}[e_1 R_{12}(t-1) + e_2 - \tag{47}$$

$$(1+\alpha) R_{12}(t-1) f[R_{12}(t-1), R_{22}(t)]]+$$

$$\theta_{222}[e_1 - f[R_{22}(t), R_{22}(t+1)]] +$$

$$\theta_{221}[e_1 - f[R_{22}(t), R_{21}(t+1)]].$$

Finally, since agents in "the market $1 \mapsto 1$" and "the market $2 \mapsto 2$" hold money and make loans, there is a set of no arbitrage conditions given by

$$R_{11}(t) = \frac{p_1(t)}{p_1(t+1)}, \tag{48}$$

and

$$R_{22}(t) = \frac{p_2(t)}{p_2(t+1)}. \tag{49}$$

In addition,

$$R_{12}(t) \geq \frac{p_1(t)}{p_2(t+1)}, \tag{50}$$

and

$$R_{21}(t) \geq \frac{p_2(t)}{p_1(t+1)}. \tag{51}$$

must hold since middle-aged agents in "the market $1 \mapsto 2$" and "the market $2 \mapsto 1$" have the option of holding money, but do not exercise that option in equilibrium.

3.3.2. Steady states

In a steady state of a Case 2 economy, $p_1(t) = p_1(t+1)$ and $p_2(t) = p_2(t+1)$ must hold. Hence $R_{11}(t) = 1 = R_{22}(t)$ must be satisfied. It follows that equation (44), in a steady state, reduces to

$$\begin{aligned} \theta_{112}\left[e_1 + e_2 - (1+\alpha) f(1, R_{12})\right] + \\ \theta_{212}\left[e_1 R_{21} + e_2 - (1+\alpha) R_{21} f(R_{21}, R_{12})\right] + \\ \theta_{121}\left[e_1 - f(R_{12}, R_{21})\right] + \theta_{122}\left[e_1 - f(R_{12}, 1)\right] \; = \; 0. \end{aligned} \tag{52}$$

Similarly, equation (45) takes the form

$$\begin{aligned} \theta_{112}\left[e_1 + e_2 - (1+\alpha) f(1, R_{21})\right] + \\ \theta_{212}\left[e_1 R_{12} + e_2 - (1+\alpha) R_{12} f(R_{12}, R_{21})\right] + \\ \theta_{121}\left[e_1 - f(R_{21}, R_{12})\right] + \theta_{122}\left[e_1 - f(R_{21}, 1)\right] \; = \; 0, \end{aligned} \tag{53}$$

where we have used our symmetric itineraries assumption in writing (53). Notice that equations (52) and (53) are completely symmetric. It is therefore natural to seek a steady state with $R_{12} = R_{21} = R$.

With this condition imposed, equations (52) and (53) reduce to the common steady state equilibrium condition

$$\begin{aligned} \theta_{112}\left[e_1 + e_2 - (1+\alpha) f(1, R)\right] + \\ \theta_{212}\left[e_1 R + e_2 - (1+\alpha) R f(R, R)\right] + \\ \theta_{121}\left[e_1 - f(R, R)\right] + \theta_{122}\left[e_1 - f(R, 1)\right] \; = \; 0. \end{aligned} \tag{54}$$

It is easy to show that the left-hand side of equation (54) is strictly decreasing in R. The following result is then immediate from the fact that (41) fails in a steady state with $R_{12} = R_{21} = 1$.

Proposition 3.1. *There is a unique monetary steady state with $R_{12} = R_{21} = R > 1$.*

Once R is determined by equation (54), equations (46) and (47) give the steady state price levels:

$$\begin{aligned} p_i \;=\; & M_i\{\theta_{111}\left[e_1+e_2-(1+\alpha)\,f\,(1,1)\right]+ \\ & \theta_{211}\left[e_1R+e_2-(1+\alpha)\,Rf\,(R,1)\right]+ \\ & \theta_{111}\left[e_1-f\,(1,1)\right]+\theta_{112}\left[e_1-f\,(1,R)\right]\}^{-1}, \end{aligned} \tag{55}$$

where, of course, our symmetric itineraries assumption has been brought to bear in the determination of p_2. It is easy to show that the right hand side of (55) is increasing in R. Moreover, if (41) fails when $R_{ij}=1$, $\forall\, i,j$, then our assumption that the centralized economy is Samuelsonian implies that

$$\begin{aligned} & \theta_{111}\left[e_1+e_2-(1+\alpha)\,f\,(1,1)\right]+ \\ & \theta_{211}\left[e_1+e_2-(1+\alpha)\,f\,(1,1)\right]+ \\ & \theta_{111}\left[e_1-f\,(1,1)\right]+\theta_{112}\left[e_1-f\,(1,1)\right] \;>\; 0 \end{aligned} \tag{56}$$

must be satisfied. Then, since $R>1$, the price level in each location must be positive. We therefore have the following result.

PROPOSITION 3.2. *Consider a Case 2 economy under a restriction against private circulating liabilities. If the centralized analog economy is Samuelsonian, outside money has value in each location in any symmetric steady state.*

It remains to check whether the conditions (50) and (51) are satisfied or not. Since (55) implies that $p_1/p_2 = M_1/M_2 = M_1/\left(M-M_1\right)$, we have the following result.

PROPOSITION 3.3. *A Case 2 economy has a symmetric steady state with outside money being valued iff*

$$R \geq \max\left\{\frac{M_1}{M-M_1}, \frac{M-M_1}{M_1}\right\}. \tag{57}$$

In particular, if (57) holds (fails), middle-aged agents traveling between locations will (will not) prefer holding one-period loans to holding outside money. Thus if (57) holds (fails) there is (is not) a symmetric steady state of the type constructed.

Notice that the exsistence of a steady state with valued outside money depends critically on the initial distribution of fiat money across locations. If the supply of outside money is equally distributed ($M_1 = M/2$), then a symmetric steady state exists. However, if the initial distribution of outside money is too unequal, there will be no such steady state [because (57) will be violated]. It was a common complaint in early U.S. monetary history that the supply of outside money was distributed in a very unequal way between east and west or north and south.[16] Thus the possibility that the outside money stock might be very unequally distributed—and that this might be problematic—should not be dismissed.

3.3.3. Local dynamics

Equations (44)-(49) constitute a system of six difference equations in $R_{11}(t)$, $R_{12}(t)$, $R_{21}(t)$, $R_{22}(t)$, $p_1(t)$, and $p_2(t)$. We again calculated local dynamics via numerical methods. The nine independent parameters of the system (under our symmetric itineraries assumption) were chosen using methods described earlier (in Section 2.5.5) in order to again create a large set of economies, each one differently parameterized. Some sets of parameter values are not consistent with a Case 2 equilibrium, and in those cases we discarded that economy and chose another one. We continued this process until we were left with $1,000$ Case 2 economies. We then constructed and verified the monetary steady state using Proposition 3.1. Next, we linearized the dynamical system and evaluated the resulting Jacobian matrix at the monetary steady state. It is easy to verify that the dimension of the system is ten, and there are now only four given initial conditions.

The results are as follows. For each Case 2 economy, there are six eigenvalues outside the unit circle and four eigenvalues inside the unit circle. Therefore, equilibrium is determinate. The eigenvalues can be complex and/or real and negative, so that equilibrium sequences may oscillate en route to the steady state. However, the absence of an eigenvalue equal to -1 means that the nature of this volatility is much less persistent than that in the economy with no prohibition on private circulating liabilities discussed earlier.

3.3.4. Welfare consequences

Under our symmetric itineraries assumption, a combination of private and public circulating liabilities supports a steady state with $R_{ij} = 1\ \forall$

[16] See Hammond (1957).

i, j. Since the golden rule allocation is attained in such a steady state, it is not possible that a prohibition against private circulating liabilities can make all agents better off, in a steady state. Moreover, any agents whose welfare is reduced by the prohibition of private circulating liabilities *cannot* be compensated by agents who gain as a result of such a prohibition. In a Case 2 economy, then, there is an argument to be made that prohibiting the issue of private circulating liabilities results in an inefficiency, relative to the monetary steady state. In addition, at least with respect to monetary steady states, the prohibition of private circulating liabilities does not eliminate any indeterminacies, and endogenously arising volatility will still be observed generically. Thus the case for prohibiting the issue of private currency seems very weak.

Nonetheless, in a comparison of steady state allocations, a *majority* of agents may *prefer* to prohibit the use of private circulating liabilities. It is therefore very possible that considerations of political economy will lead to a monetary arrangement where private circulating liabilities are prohibited, even though the economic arguments seem strongly against such a prohibition. We now show how this can occur. We begin with a discussion of which agents gain and which agents lose, in a comparison of monetary steady states with and without private circulating liabilities.

For agents with the itineraries $(1,1,1)$ and $(2,2,2)$, utility is $V(1,1)$ in a monetary steady state whether private liabilities circulate or not. Thus these agents are indifferent regarding a prohibition against private circulating liabilities.

For agents with the itineraries $(1,1,2)$ and $(2,2,1)$, their lifetime utility in a monetary steady state where private liabilities circulate is $V(1,1)$. Their lifetime utility where private circulating liabilities are prohibited is $V(1,R)$. Lemma 2.1, along with the fact that $R > 1$, implies that $V(1,R) > V(1,1)$ holds. Hence these agents will strictly prefer that private circulating liabilities be prohibited.

Agents having the itineraries $(1,2,1)$ and $(2,1,2)$ have the lifetime utility level $V(1,1)$ in a monetary steady state with public and private circulating liabilities. When private circulating liabilities are prohibited, their utility is $V(R,R)$. Lemma 2.1 and $R > 1$ imply that $V(R,R) > V(1,1)$, if $H(1) > 0$ and $\beta \leq 1$ hold. Since $H(1) > 0$ is necessarily satisfied in a Samuelson case economy, agents who relocate twice also prefer that private circulating liabilities be prohibited, if $\beta \leq 1$.

The only agents who are harmed by a prohibition on private circulating liabilities are those with the itineraries $(1,2,2)$ and $(2,1,1)$. In a monetary steady state with circulating private liabilities, their utility is $V(1,1)$. If private liabilities do not circulate, their utility is $V(R,1)$. Since $R>1$, Lemma 2.1 implies that these agents will be worse off if private liabilities cannot be employed.

To summarize, then, in a comparison of steady state allocations, two of the four possible groups of agents strictly prefer the prohibition of private circulating liabilities. One group is indifferent, and one group is strictly harmed by such a prohibition.

Using our symmetry assumption, the mass of agents that strictly prefer to prohibit the use of private circulating liabilities is $\theta_{112}+\theta_{121}$ (in each location). The mass of agents that is strictly opposed to such a prohibition is θ_{122} (in each location). There remains the question of whether $\theta_{112}+\theta_{121}>\theta_{122}$ can hold, while at the same time parameter configurations leave us in a Case 2 economy. Appendix E demonstrates that such an outcome can indeed occur. In fact, as the appendix shows, it is possible to set $\theta_{112}+\theta_{121}>\theta_{111}+\theta_{122}$, consistent with this being a Case 2 economy. We can conclude, then, that it is quite possible for political economy considerations to lead to a prohibition of private circulating liabilities, despite the lack of a strong economic argument against such a prohibition. This is consistent with the observation that prohibitions against, or general restrictions on the use of private circulating liabilities have been very common historically.

3.3.5. Discounts on notes

Clearly a prohibition against private circulating liabilities causes them to be in zero net supply. Nonetheless, we can calculate the implied discounts on privately-issued circulating media of exchange.[17] In particular, suppose that at date t, some (middle-aged) agent were to purchase a previously-issued, private circulating liability in location 1—say a claim to \$1 of outside money in location i at $t+1$. The real value of this claim at $t+1$ would be $p_i(t+1)^{-1}$, and the discounted present value of this claim would be $1/p_i(t+1)R_{1i}(t)$ in location 1 at date t. Finally, the nominal value of this claim in location 1 at t would be $p_1(t)/p_i(t+1)R_{1i}(t)$. Note that, if $i=1$, the claim would sell for \$1; that is, it would *not* be discounted relative to outside money, in its location of origin. However, if $i=2$, then $p_1(t)/p_i(t+1)R_{1i}(t)=p_1/p_2R<1$ can hold, and the notes of private agents will be discounted. How large will the implied discount be? The answer depends on the imbalance between regional money supplies. Since

[17] In practice, it was less the case that private note issue was prohibited in the early 19^{th} century than that there were restrictions on the quantities and denominations of these notes that could be issued. We hope that our analysis sheds light on this situation as well as the one where private circulating liabilities are prohibited.

$p_1/p_2 = M_1/(M - M_1)$,[18] discounts on private notes will be small in regions where the supply of outside money is fairly large.

As a practical matter, it has proven difficult to understand the historically observed discounts on privately-issued notes by appeal to obvious factors such as redemption costs or default risks on notes (see Gorton [1989]). This analysis suggests why that should be the case. It suggests instead that the relative stocks of outside money in different regions are likely to be important factors in determining discounts on privately-issued notes, at least when there are some legal restrictions on private note issue.

3.4. A Case 3 economy

Under our symmetric itineraries assumption, a Case 3 economy will emerge under a prohibition against private circulating liabilities if

$$(\theta_{111} + \theta_{122})\left[e_1 + e_2 - (1+\alpha) f(1,1)\right] + (\theta_{111} + \theta_{112})\left[e_1 - f(1,1)\right] < 0 \tag{58}$$

and

$$(\theta_{112} + \theta_{121})\left[e_1 + e_2 - (1+\alpha) f(1,1)\right] + (\theta_{121} + \theta_{122})\left[e_1 - f(1,1)\right] > 0 \tag{59}$$

hold. Inequalities (58) and (59) assert that there is an excess demand for credit in "the market $1 \mapsto 1$," and an excess supply of credit in "the market $1 \mapsto 2$" (and similarly for "the markets $2 \mapsto 2$ and $2 \mapsto 1$"), when $R_{11} = R_{12} = R_{21} = R_{22} = 1$. Under these circumstances, outside money will be held by middle-aged agents who *change* location; that is, in each period the stock of outside money that circulates in location 1 at t is carried to location 2 at $t+1$ and conversely.

This variability in regional money supplies means that

$$M_1(t) = \begin{cases} M_1, & t \text{ even} \\ M - M_1, & t \text{ odd} \end{cases} \tag{60}$$

where M_1 is the given initial stock of outside money in location 1. And, so long as $M_1 \neq M/2$, it is reasonable to conjecture that the variability in the stock of outside money circulating in each region will lead to some economic volatility. While this conjecture is correct, as we now establish, we also show that it is possible that the economic consequences of this volatility are quite minimal, being confined to volatility in the price level.

[18] And, since R is independent of the regional allocation of outside money.

In order to establish our claim, we seek an equilibrium where

$$p_i(t) = \begin{cases} p_i^e, & t \text{ even} \\ p_i^o, & t \text{ odd} \end{cases} \tag{61}$$

where $p_1^o = p_2^e$ and $p_1^e = p_2^o$.[19] Then, since middle-aged agents carry outside money between locations at each date, $R_{12} = p_1^e/p_2^o = 1/R_{21} = 1$ holds in a periodic equilibrium of the type sought. In addition, we conjecture the existence of an equilibrium where $R_{11} = R_{22} = R$ at all dates. Then the supply of credit equals the demand for credit in "the markets $1 \mapsto 1$ and $2 \mapsto 2$" if

$$\begin{aligned} &\theta_{111}\left[e_1 R + e_2 - (1+\alpha) R f(R,R)\right] + \\ &\theta_{122}\left[e_1 + e_2 - (1+\alpha) f(1,R)\right] + \\ &\theta_{111}\left[e_1 - f(R,R)\right] + \theta_{112}\left[e_1 - f(R,1)\right] = 0 \end{aligned} \tag{62}$$

It is straightforward to show that the left-hand side of (62) is increasing in R. Therefore (58) implies that $R > 1$. In addition, "the market $1 \mapsto 2$" clears at t if

$$\begin{aligned} \tfrac{M_1(t)}{p_1(t)} &= \theta_{112}\left[e_1 R + e_2 - (1+\alpha) R f(R,1)\right] + \\ &\quad \theta_{121}\left[e_1 + e_2 - (1+\alpha) f(1,1)\right] + \\ &\quad \theta_{121}\left[e_1 - f(1,1)\right] + \theta_{122}\left[e_1 - f(1,R)\right]. \end{aligned} \tag{63}$$

And, "the market $2 \mapsto 1$" clears at t (using our symmetric itineraries assumption) if

$$\begin{aligned} \tfrac{M - M_1(t)}{p_2(t)} &= \theta_{112}\left[e_1 R + e_2 - (1+\alpha) R f(R,1)\right] + \\ &\quad \theta_{121}\left[e_1 + e_2 - (1+\alpha) f(1,1)\right] + \\ &\quad \theta_{121}\left[e_1 - f(1,1)\right] + \theta_{122}\left[e_1 - f(1,R)\right]. \end{aligned} \tag{64}$$

[19] The fact that the money stock in location 1 in even periods is transferred to location 2 in odd periods and vice-versa, along with our symmetric itineraries assumption, implies that location 1 in even (odd) periods has the same characteristics as location 2 in odd (even) periods.

But (63), (64), and $M_1(t+1) = M - M_1(t)$ imply that $p_1(t+1) = p_2(t)$, as desired. Moreover, the no-arbitrage condition

$$R \geq \max\left\{\frac{p_1^e}{p_2^o}, \frac{p_2^o}{p_1^e}\right\} \tag{65}$$

is always satisfied, in the equilibrium we have constructed. Thus an equilibrium with the desired properties necessarily exists.

Thus, in a Case 3 economy, there will be regional price level fluctuations, as well as fluctuations in note discounts. These fluctuations will *not* affect the steady state allocation of resources, and hence are innocuous.

Finally, we note that the same economic arguments apply against the prohibition of private circulating liabilities as applied earlier. Nonetheless, calculations similar to those conducted earlier can be used to show that political economy considerations may lead to such a prohibition.

3.5. A comment on asymmetric itineraries

When we relax our symmetry assumption, then it is possible, for example, that there will be excess demand for credit in "the market $1 \mapsto 2$" and "the market $2 \mapsto 2$." As a result, no outside money will ever be carried from location 1 to location 2, and no outside money will ever remain in location 2. In short, within one period all outside money can reside in location 1. Thus, in the absence of symmetry, one region can be monetary while the other is not.

Hammond (1957) discusses the severe perceived imbalances of interregional money supplies and money flows during early American monetary history. And, Jackson's "specie circular" has often been charged with denuding the western U.S. of currency. While historians have tended to downplay the possible importance of these assertions, our analysis suggests that these could have been real—as well as costly—phenomena.

4. MORE GENERAL ITINERARIES AND THE ROLE FOR OUTSIDE MONEY

Under our symmetric itineraries condition, outside money had value in the decentralized economy (with circulating private liabilities permitted) if and only if it had value in the centralized economy. Or, in other words, a decentralized economy needed to have a Samuelsonian centralized analog in order for outside money to have value.

When we allow for more general itineraries, this is no longer the case. Indeed, the dual frictions of spatial separation and limited communication can either enhance or reduce the role for outside money. In particular, we will demonstrate that decentralized economies whose centralized analogs are Samuelsonian may admit *no* role for outside money. Thus spatial separation and limited communication can have quite complicated consequences for the role of money in an economy.

In addition, under our symmetric itineraries assumption, a combination of inside and outside money could—at least with reference to steady states—completely overcome the frictions implied by spatial separation and limited communication. As we will see, this is no longer generally true when we depart from symmetric itineraries. Thus, in general economies, public and private circulating liabilities alone are not adequate to deal with the frictions of spatial separation and limited communcation.

In order to illustrate the latter point, recall that an equilibrium with general itineraries is a set of sequences $\{R_{11}(t)\}$, $\{R_{12}(t)\}$, $\{R_{21}(t)\}$, and $\{R_{22}(t)\}$ satisfying the no-arbitrage conditions (16) and (20), and the market clearing conditions (25) and (30). For the moment we focus on monetary steady states, and establish the following result.

PROPOSITION 4.1. *Part (a). There exists a steady state solution of equations (16), (20), (25), and (30) with* $R_{11} = R_{22} = 1$, $R_{21} = 1/R_{12}$, *and*

$$\begin{aligned} R_{12} = {} & \{\theta_{211}(\alpha+\beta)e_1 + \theta_{212}\alpha(e_1+e_3) + \\ & \theta_{221}\beta(e_1+e_3) + \theta_{112}(1+\alpha)e_3 + \\ & \theta_{121}(1+\beta)e_2 + \theta_{122}(e_2+e_3)\}\times \\ & \{\theta_{122}(\alpha+\beta)e_1 + \theta_{121}\alpha(e_1+e_3) + \\ & \theta_{112}\beta(e_1+e_2) + \theta_{221}(1+\alpha)e_3 + \\ & \theta_{212}(1+\beta)e_2 + \theta_{211}(e_2+e_3)\}^{-1} \end{aligned} \tag{66}$$

Part (b). $R_{12} = 1$ *holds iff*

$$\begin{aligned} & (\theta_{221}-\theta_{112})\left[\left(\tfrac{\beta}{1+\alpha+\beta}\right)(e_1+e_2+e_3) - e_3\right] + \\ & (\theta_{212}-\theta_{121})\left[\left(\tfrac{\alpha}{1+\alpha+\beta}\right)(e_1+e_2+e_3) - e_2\right] - \\ & (\theta_{211}-\theta_{122})\left[\left(\tfrac{1}{1+\alpha+\beta}\right)(e_1+e_2+e_3) - e_1\right] = 0 \end{aligned} \tag{67}$$

Proof. See Appendix F. ∎

Note that our symmetric itineraries assumption implies that (67) is satisfied, and hence that $R_{12} = R_{21} = 1$. However, when (67) fails—as it will generically—then $R_{11} = R_{22} = 1 \neq R_{12} \neq R_{21}$. Thus different agents face different rates of return, depending on their itineraries. As a consequence, a simple combination of public and private circulating liabilities cannot completely overcome the frictions implied by spatial separation and limited communication. And, how close they can come to doing so depends on how "close" the fractions of agents following different itineraries are to being symmetric.

For completeness, we note that extensive numerical analysis revealed that there is always one other "non-monetary" steady state in this system. Local dynamics in a neighborhood of either steady state are exactly as described in the case of symmetric itineraries.

One final question remains. Under what circumstances will there be a steady state in which outside money has value in a decentralized economy with general intineraries on the part of agents? Providing a complete answer to this question is beyond the scope of this paper. But we do wish to illustrate the following two points.

Claim 4.1. There exist decentralized economies whose centralized analogs are classical, but where outside money is valued (in both locations) in a monetary steady state equilibrium.

Claim 4.2. There exist decentralized economies whose centralized analogs are Samuelsonian, but which have no steady state equilibrium where outside money is valued.

Proof. Both claims are proved in Appendix G. ∎

As these claims illustrate, the consequences of spatial separation and limited communication for the role of outside money can be quite complicated. Put simply, this is because—with general itineraries—spatial separation and limited communication have quite complicated implications for savings behavior. Hence, relative to its centralized analog, a decentralized economy can have either a higher or lower propensity to save, and nonmonetary economies can be converted into monetary economies and vice versa.

5. CONCLUSION

We have examined an economy where spatial separation and limited communication imply that trade will take place among agents in a variety of distinct markets. Moreover, the endowment patterns of agents and the patterns of agents' itineraries may imply that some agents wish to issue liabilities to other agents who they will never meet again. These liabilities must, therefore, circulate. And, they may circulate alongside a stock of government-issued, or "outside," money.

In this context we have examined several different monetary arrangements. Some consist of a mixture of publicly- and privately-issued, circulating liabilities. Others consist entirely of purely public, or of purely private circulating liabilities. In an economy whose centralized analog is Samuelsonian, the attainment of an efficient allocation of resources requires the presence of outside money. Here Hayekian proposals for systems where money creation is "left to the market" are not consistent with an optimal allocation of resources. Moreover, if the fraction of agents following different itineraries shows sufficient diversity, systems of purely public liabilities are also inconsistent with the attainment of an optimal resource allocation. Under these circumstances, full efficiency cannot be attained unless both public and private entities issue "money-like" liabilities.

Moreover, we have shown that a prohibition against private circulating liabilities—of the type advocated, for instance, by Friedman (1960)—should not be expected to reduce the set of equilibria nor to reduce the scope for endogenously arising volatility. Indeed, if the stock of outside money is too unevenly distributed across locations, such a prohibition may preclude the existence of any steady state equilibrium whatsoever. Thus, in our economy, the real bills doctrine has considerable force. Indeed, there is a strong case to be made for allowing the co-existence of publicly- and privately-issued currencies.

Nonetheless, we have demonstrated that considerations of political economy may lead to a prohibition against private circulating liabilities. Such considerations may help explain why prohibitions have been commonplace historically, despite the strong economic arguments against them.

Finally, we have shown that, when there is considerable asymmetry in agents' patterns of movement, spatial separation and limited communication can have complicated consequences regarding the role for outside money to circulate. Indeed, as we have shown, spatial separation and limited communication can permit outside money to circulate in an economy whose centralized analog does not admit outside money. However, the reverse is also true. Spatial separation and limited communication can eliminate a role for outside money, even if outside money could circulate in an economy's centralized analog. This is an important point, as it implies that improvements in communication need not imply a declining role for outside money, as many have argued.

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APPENDIX A
Proof of Lemma 2.1

From the definition of $V\left[R_{hi}(t), R_{ij}(t+1)\right]$, we have

$$V\left[R_{hi}(t), R_{ij}(t+1)\right] = (1+\alpha+\beta)\ln\left\{f\left[R_{hi}(t), R_{ij}(t+1)\right]\right\} + \quad \text{(A.1)}$$

$$(\alpha+\beta)\ln R_{hi}(t) + \beta\ln R_{ij}(t+1) + \alpha\ln\alpha + \beta\ln\beta.$$

Part (a). Differentiating (A.1) and rearranging terms yields

$$R_{hi}(t)\, V_1\left[R_{hi}(t), R_{ij}(t+1)\right] = \quad \text{(A.2)}$$

$$\alpha+\beta+(1+\alpha+\beta)\frac{R_{hi}(t)\, f_1\left[R_{hi}(t), R_{ij}(t+1)\right]}{f\left[R_{hi}(t), R_{ij}(t+1)\right]}.$$

Moreover, since

$$\frac{R_{hi}(t)\, f_1\left[R_{hi}(t), R_{ij}(t+1)\right]}{f\left[R_{hi}(t), R_{ij}(t+1)\right]} = -\frac{\left[\frac{e_2}{R_{hi}(t)} + \frac{e_3}{R_{hi}(t)R_{ij}(t+1)}\right]}{e_1 + \frac{e_2}{R_{hi}(t)} + \frac{e_3}{R_{hi}(t)R_{ij}(t+1)}}, \quad \text{(A.3)}$$

it follows that $V_1 < (\geq)\ 0$ holds iff

$$(\alpha+\beta)\, e_1 < (\geq)\ \frac{e_2}{R_{hi}(t)} + \frac{e_3}{R_{hi}(t)\, R_{ij}(t+1)}. \quad \text{(A.4)}$$

But, it is easy to verify that $f\left[R_{hi}(t), R_{ij}(t+1)\right] > (\leq)\ e_1$ iff (A.4) holds.

Part (b). Again differentiating (A.1) and rearranging terms yields

$$R_{ij}(t+1)\, V_2\left[R_{hi}(t), R_{ij}(t+1)\right] = \quad \text{(A.5)}$$

$$\beta + (1+\alpha+\beta)\frac{R_{ij}(t)\, f_2[R_{hi}(t), R_{ij}(t+1)]}{f[R_{hi}(t), R_{ij}(t+1)]}.$$

Moreover,

$$\frac{R_{ij}(t+1)\, f_2[R_{hi}(t), R_{ij}(t+1)]}{f[R_{hi}(t), R_{ij}(t+1)]} = -\frac{\left[\frac{e_3}{R_{hi}(t)R_{ij}(t+1)}\right]}{e_1 + \frac{e_2}{R_{hi}(t)} + \frac{e_3}{R_{hi}(t)R_{ij}(t+1)}}. \quad \text{(A.6)}$$

It follows that $V_2 > (\leq)\ 0$ holds iff

$$\beta\left[e_1 + \frac{e_2}{R_{hi}(t)}\right] > (\leq)\ (1+\alpha)\frac{e_3}{R_{hi}(t)\, R_{ij}(t+1)}. \quad \text{(A.7)}$$

But (A.7) is equivalent to

$$c_t(t+2) = \beta R_{hi}(t)\, R_{ij}(t+1)\, f[R_{hi}(t), R_{ij}(t+1)] > (\leq)\ e_3. \quad \text{(A.8)}$$

Part (c). Define

$$\begin{aligned} \tilde{V}(R) &\equiv V(R,R) \\ &= (1+\alpha+\beta)\ln[f(R,R)] + \\ &\quad (\alpha+2\beta)\ln R + \alpha\ln\alpha + \beta\ln\beta. \end{aligned} \quad \text{(A.9)}$$

Then

$$\begin{aligned} R\tilde{V}'(R) &= \alpha + 2\beta - \\ &\quad (1+\alpha+\beta)\left[\frac{Rf_1(R,R)+Rf_2(R,R)}{e_1+\frac{e_2}{R}+\frac{e_3}{R^2}}\right]. \end{aligned} \quad \text{(A.10)}$$

Thus $\tilde{V}'(R) > (\leq)\ 0$ holds iff

$$0 < (\geq)\ (\alpha+2\beta)e_1 + (\beta-1)\frac{e_2}{R} - (\alpha+2)\frac{e_3}{R^2} \equiv H(R). \quad \text{(A.11)}$$

Note that, in an economy whose centralized analog is Samuelsonian, $H(1) > 0$. Moreover, $H(R) > 0\ \forall\ R > 1$ if $\beta \leq 1$.

APPENDIX B
Proof of Lemma 2.2

At the monetary steady state, the characteristic polynomial of J takes the form

$$\lambda^3 - \lambda^2\frac{\partial R(t+1)}{\partial R(t)} - \lambda\frac{\partial R(t+1)}{\partial u(t)} - \frac{\partial R(t+1)}{\partial v(t)} = 0. \quad \text{(B.1)}$$

It is also straightforward to verify that, at the monetary steady state,

$$\frac{\partial R(t+1)}{\partial R(t)} = -\alpha - \frac{f_1(1,1)}{f_2(1,1)}, \tag{B.2}$$

$$\frac{\partial R(t+1)}{\partial u(t)} = -\beta - \frac{\alpha f_1(1,1)}{f_2(1,1)} - (\alpha+\beta)\frac{f(1,1)}{f_2(1,1)}, \tag{B.3}$$

and

$$\frac{\partial R(t+1)}{\partial v(t)} = -\beta\left[\frac{f_1(1,1)}{f_2(1,1)}\right] - \beta\left[\frac{f(1,1)}{f_2(1,1)}\right]. \tag{B.4}$$

In addition,

$$\frac{f_1(1,1)}{f_2(1,1)} = \frac{e_2+e_3}{e_3} \tag{B.5}$$

and

$$\frac{f(1,1)}{f_2(1,1)} = -\left(\frac{e_1+e_2+e_3}{e_3}\right). \tag{B.6}$$

It follows that the characteristic polynomial of J can be written as

$$\lambda^3 + \lambda^2\left[(1+\alpha) + \left(\frac{e_2}{e_3}\right)\right] - \lambda\left[(\alpha+\beta)\left(\frac{e_1}{e_3}\right) + \beta\left(\frac{e_2}{e_3}\right)\right] \tag{B.7}$$

$$-\beta\left(\frac{e_1}{e_3}\right) \equiv J(\lambda) = 0.$$

Evidently,

$$\lim_{\lambda\to-\infty} J(\lambda) = -\infty, \tag{B.8}$$

$$\lim_{\lambda\to\infty} J(\lambda) = \infty, \tag{B.9}$$

and

$$J(0) < 0 \tag{B.10}$$

all hold. Moreover,

$$J(1) = \frac{(2+\alpha)e_3 - (\beta-1)e_2 - (\alpha+2\beta)e_1}{e_3} < 0 \tag{B.11}$$

in a Samuelson case economy. And

$$J(-1) = \frac{(1+\beta)e_2 + \alpha(e_1+e_3)}{e_3} > 0. \tag{B.12}$$

Thus J has one eigenvalue in the interval $(-\infty,-1)$, one eigenvalue in the interval $(-1,0)$, and one eigenvalue in the interval $(1,\infty)$, as claimed.

APPENDIX C
Proof of Lemma 2.3

Equations (16) and (17) are equivalent to

$$R_{11}(t)\, R_{11}(t+1) = R_{12}(t)\, R_{21}(t+1) \tag{C.1}$$

for all $t \geq 0$, and

$$\frac{R_{11}(t+1)}{R_{12}(t+1)} = \frac{R_{21}(t+1)}{R_{22}(t+1)}. \tag{C.2}$$

for all $t \geq 0$. Similarly, equations (20) and (21) are equivalent to equation (C.2) and

$$R_{22}(t)\, R_{22}(t+1) = R_{21}(t)\, R_{12}(t+1)\,, \tag{C.3}$$

for all $t \geq 0$. Moreover, by (24),

$$\frac{R_{11}(0)}{R_{12}(0)} = \frac{p_1(0)}{p_1(1)} \frac{p_2(1)}{p_1(0)} = \frac{p_2(0)}{p_1(1)} \frac{p_2(1)}{p_2(0)} = \frac{R_{21}(0)}{R_{22}(0)}. \tag{C.4}$$

Now note that (16) and (20) imply that

$$\frac{R_{11}(t)\, R_{11}(t+1)}{R_{21}(t)\, R_{12}(t+1)} = \frac{R_{12}(t)\, R_{21}(t+1)}{R_{22}(t)\, R_{22}(t+1)} \tag{C.5}$$

for all $t \geq 0$. But conditions (C.4) and (C.5) imply (C.2). Thus equations (16), (20), and (24) imply all of the remaining no-arbitrage conditions, as claimed.

APPENDIX D
Proof of Proposition 2.1

In a steady state, the no-arbitrage conditions imply that $R_{11} = R_{22} = R$, and that $R_{12}R_{21} = R^2$. It follows that the location 1 goods market clearing condition, (25), can be written as

$$\begin{gathered}\theta_{111}\left(1+\alpha R+\beta R^2\right) f(R, R)+\theta_{112}(1+\alpha R) f\left(R, R_{12}\right)+ \\ \theta_{121}\left(1+\beta R^2\right) f\left(R_{12}, R_{21}\right)+\theta_{122} f\left(R_{12}, R\right)+\theta_{212} \alpha R_{21} f\left(R_{21}, R_{12}\right)+ \\ \theta_{211} \alpha\left(R_{21}+\beta R_{21} R\right) f\left(R_{21}, R\right)+\theta_{221} \beta R R_{21} f(R, R)- \\ \left(\theta_{111}+\theta_{112}+\theta_{121}+\theta_{122}\right) e_1-\left(\theta_{111}+\theta_{112}+\theta_{212}+\theta_{211}\right) e_2- \\ \left(\theta_{111}+\theta_{121}+\theta_{211}+\theta_{221}\right) e_3=0.\end{gathered} \tag{D.1}$$

Similarly, given the symmetric itineraries assumption, the location 2 goods market clearing condition takes the form

$$\begin{gathered}\theta_{111}\left(1+\alpha R+\beta R^2\right) f(R, R)+\theta_{112}(1+\alpha R) f\left(R, R_{21}\right)+ \\ \theta_{121}\left(1+\beta R^2\right) f\left(R_{21}, R_{12}\right)+\theta_{122} f\left(R_{21}, R\right)+\theta_{212} \alpha R_{12} f\left(R_{12}, R_{21}\right)+ \\ \theta_{211}\left(\alpha R_{12}+\beta R R_{12}\right) f\left(R_{12}, R\right)+\theta_{221} \beta R R_{12} f\left(R, R_{12}\right)- \\ \left(\theta_{111}+\theta_{112}+\theta_{121}+\theta_{122}\right) e_1-\left(\theta_{111}+\theta_{112}+\theta_{212}+\theta_{211}\right) e_2- \\ \left(\theta_{111}+\theta_{121}+\theta_{211}+\theta_{221}\right) e_3=0.\end{gathered} \tag{D.2}$$

Obviously, if $R_{12} = R_{21} = R$, the no-arbitrage conditions are satisfied. Moreover, satisfaction of (D.1) implies satisfaction of (D.2). Thus there exist steady states with $R_{12} = R_{21} = R$, where R satisfies

$$\left(\theta_{111}+\theta_{112}+\theta_{121}+\theta_{122}\right)\left(e_1+e_2+e_3\right)- \tag{D.3}$$

$$\left(\theta_{111}+\theta_{112}+\theta_{121}+\theta_{122}\right) f(R, R)\left(1+\alpha R+\beta R^2\right)=0.$$

In particular, (D.3) is obtained by setting $R_{12} = R_{21} = R$ in (D.1), and using the symmetric itineraries assumption. Obviously (D.3) is equivalent to

$$\left(1+\alpha R+\beta R^2\right) f(R, R)=e_1+e_2+e_3. \tag{D.4}$$

But (D.4) is exactly the steady state equilibrium condition for the centralized economy. This establishes Proposition 2.1.

APPENDIX E
Proof of Proposition 2.2

Recall that, in order to have a Case 2 economy, we must have an "excess supply" of credit in "the market $1 \mapsto 1$," and an "excess demand" for credit in "the market $1 \mapsto 2$" when $R_{ij} = 1$, $\forall\ i, j$. This requires that

$$(\theta_{111} + \theta_{122})\left[e_1 + e_2 - (1+\alpha) f(1,1)\right] + \tag{E.1}$$

$$(\theta_{111} + \theta_{112})\left[e_1 - f(1,1)\right] > 0 >$$

$$(\theta_{112} + \theta_{121})\left[e_1 + e_2 - (1+\alpha) f(1,1)\right] +$$

$$(\theta_{121} + \theta_{122})\left[e_1 - f(1,1)\right] =$$

$$(\theta_{112} + \theta_{121})\left\{\left[e_1 + e_2 - (1+\alpha) f(1,1)\right] + \left[e_1 - f(1,1)\right]\right\}$$

$$+ (\theta_{122} - \theta_{112})\left[e_1 - f(1,1)\right].$$

In addition, in order to have a Samuelson case economy with credit demand by young agents, we must have

$$\left[e_1 + e_2 - (1+\alpha) f(1,1)\right] + \left[e_1 - f(1,1)\right] > 0 > e_1 - f(1,1). \tag{E.2}$$

It is straightforward to show that the inequalities in (E.1) can be rearranged to yield

$$\left\{\left[e_1 + e_2 - (1+\alpha) f(1,1)\right] + \left[e_1 - f(1,1)\right]\right\}\theta_{111} - \left[f(1,1) - e_1\right]\theta_{112} > \tag{E.3}$$

$$- \left[e_1 + e_2 - (1+\alpha) f(1,1)\right]\theta_{122}$$

and

$$\theta_{122} > \left[\frac{e_1 + e_2 - (1+\alpha) f(1,1)}{f(1,1) - e_1}\right]\theta_{112} + \tag{E.4}$$

$$\left\{\frac{\left[e_1 + e_2 - (1+\alpha) f(1,1)\right] - \left[f(1,1) - e_1\right]}{f(1,1) - e_1}\right\}\theta_{121}.$$

Conditions (E.3) and (E.4) suffice to make this a Case 2 economy.

We now observe that the inequality $\theta_{112} + \theta_{121} > \theta_{122}$ is consistent with the satisfaction of (E.3) and (E.4) if

$$\left\{\left[e_1 + e_2 - (1+\alpha) f(1,1)\right] + \left[e_1 - f(1,1)\right]\right\}\theta_{111} - \left[f(1,1) - e_1\right]\theta_{112} > \tag{E.5}$$

$$-\left[e_1+e_2-(1+\alpha)\,f(1,1)\right](\theta_{112}+\theta_{121})$$

and if

$$\theta_{112}+\theta_{121} > \left[\frac{e_1+e_2-(1+\alpha)\,f(1,1)}{f(1,1)-e_1}\right]\theta_{112}+ \tag{E.6}$$

$$\left\{\frac{\left[e_1+e_2-(1+\alpha)\,f(1,1)\right]-\left[f(1,1)-e_1\right]}{f(1,1)-e_1}\right\}\theta_{121}.$$

Rearranging terms in (E.5) and (E.6) gives us the equivalent conditions

$$\left\{\left[e_1+e_2-(1+\alpha)\,f(1,1)\right]+\left[e_1-f(1,1)\right]\right\}\theta_{111}+ \tag{E.7}$$

$$\left\{\left[e_1+e_2-(1+\alpha)\,f(1,1)\right]+\left[e_1-f(1,1)\right]\right\}\theta_{112}$$

$$> -\left[e_1+e_2-(1+\alpha)\,f(1,1)\right]\theta_{121}$$

and

$$\left\{2\left[f(1,1)-e_1\right]-\left[e_1+e_2-(1+\alpha)\,f(1,1)\right]\right\}\theta_{121} > \tag{E.8}$$

$$\left[e_1+e_2-(1+\alpha)\,f(1,1)\right]\theta_{112}.$$

Under our assumptions, (E.7) necessarily holds. Moreover, it is possible to choose θ_{111}, θ_{112}, θ_{121}, and θ_{122} consistent with the satisfaction of (E.8) so that this is a Case 2 economy, so that $\theta_{112}+\theta_{121} > \theta_{122}$, and so that $\theta_{111}+\theta_{112}+\theta_{121}+\theta_{122}=\frac{1}{2}$ all hold. In other words, if (E.8) holds we can easily make this a Case 2 economy where $\theta_{112}+\theta_{121} > \theta_{122}$. Indeed, we can make this a Case 2 economy where $\theta_{112}+\theta_{121} > \theta_{111}+\theta_{122}$ holds. Here is an example illustrating this point.

Example E.0.1. Let $e_1=.05$, $e_2=.95$, and $e_3=0$, and let $\alpha=\beta=1$. Also, let $\theta_{111}=0.15$, $\theta_{112}=0.045$, $\theta_{121}=0.5$, and $\theta_{122}=0.305$. Then it is easy to verify that (E.8) holds, along with the other conditions stated.

APPENDIX F
Proof of Proposition 5.1

Part (a). If there exists a steady state equilibrium with $R_{11}=R_{22}=1$, then the no-arbitrage conditions imply that $R_{21}=1/R_{12}$. Moreover, in this case, the goods market clearing conditions reduce to

$$\begin{aligned}
&\theta_{111}(1+\alpha+\beta) f(1,1)+\theta_{112}(1+\alpha) f(1,R_{12})+\\
&\theta_{121}(1+\beta) f(R_{12},R_{21})+\theta_{122} f(R_{12},1)+\\
&\theta_{212}\alpha R_{21} f(R_{21},R_{12})+\theta_{211}(\alpha+\beta) R_{21} f(R_{21},1)+\\
&\theta_{221}\beta R_{21} f(1,R_{21})-(\theta_{111}+\theta_{112}+\theta_{121}+\theta_{122}) e_1-\\
&(\theta_{111}+\theta_{112}+\theta_{212}+\theta_{211}) e_2-(\theta_{111}+\theta_{121}+\theta_{211}+\theta_{221}) e_3 = 0
\end{aligned} \tag{F.1}$$

and

$$\begin{aligned}
&\theta_{222}(1+\alpha+\beta) f(1,1)+\theta_{221}(1+\alpha) f(1,R_{21})+\\
&\theta_{212}(1+\beta) f(R_{21},R_{12})+\theta_{211} f(R_{21},1)+\\
&\theta_{121}\alpha R_{12} f(R_{12},R_{21})+\theta_{122}(\alpha+\beta) R_{12} f(R_{12},1)+\\
&\theta_{112}\beta R_{12} f(1,R_{12})-(\theta_{222}+\theta_{221}+\theta_{212}+\theta_{211}) e_1-\\
&(\theta_{222}+\theta_{221}+\theta_{121}+\theta_{122}) e_2-(\theta_{222}+\theta_{212}+\theta_{122}+\theta_{112}) e_3 = 0.
\end{aligned} \tag{F.2}$$

It will now prove useful to state some results about steady state consumption.

LEMMA F.0.1. *Consider a steady state with $R_{11}=R_{22}=1$ and $R_{21}=1/R_{12}$. Then*

$$f(R_{12},R_{21}) = f(1,1)+\left(\frac{1-R_{12}}{R_{12}}\right)\left(\frac{e_2}{1+\alpha+\beta}\right), \tag{F.3}$$

$$f(R_{21},R_{12}) = f(1,1)+(R_{12}-1)\left(\frac{e_2}{1+\alpha+\beta}\right), \tag{F.4}$$

$$f(1,R_{12}) = f(1,1)+\left(\frac{1-R_{12}}{R_{12}}\right)\left(\frac{e_3}{1+\alpha+\beta}\right), \tag{F.5}$$

$$f(1,R_{21}) = f(1,1)-(1-R_{12})\left(\frac{e_3}{1+\alpha+\beta}\right), \tag{F.6}$$

$$f(R_{12},1) = f(1,1)+\left(\frac{1-R_{12}}{R_{12}}\right)\left(\frac{e_2+e_3}{1+\alpha+\beta}\right), \tag{F.7}$$

and

$$f\left(R_{21},1\right)=f\left(1,1\right)-\left(1-R_{12}\right)\left(\frac{e_2+e_3}{1+\alpha+\beta}\right). \tag{F.8}$$

Proof. These facts follow from the definition of the function f, along with $R_{12}R_{21}=1$. ▮

Using Lemma F.0.1, we can rewrite the market clearing conditions as

$$\begin{gathered}
f\left(1,1\right)\left\{\theta_{111}\left(1+\alpha+\beta\right)+\theta_{112}\left(1+\alpha\right)+\theta_{121}\left(1+\beta\right)+\right.\\
\left.\theta_{122}+\theta_{212}\alpha+\theta_{211}\left(\alpha+\beta\right)+\theta_{221}\beta\right\}+\\
\left(\tfrac{1-R_{12}}{R_{12}}\right)f\left(1,1\right)\left\{\theta_{212}\alpha+\theta_{211}\left(\alpha+\beta\right)+\theta_{221}\beta\right\}+\\
\left(\tfrac{1-R_{12}}{R_{12}}\right)\left\{\theta_{112}\left(\tfrac{1+\alpha}{1+\alpha+\beta}\right)e_3+\theta_{121}\left(\tfrac{1+\beta}{1+\alpha+\beta}\right)e_2+\theta_{122}\left(\tfrac{e_2+e_3}{1+\alpha+\beta}\right)-\right.\\
\left.\theta_{212}\left(\tfrac{\alpha}{1+\alpha+\beta}\right)e_2-\theta_{211}\left(\tfrac{\alpha+\beta}{1+\alpha+\beta}\right)\left(\tfrac{e_2+e_3}{1+\alpha+\beta}\right)-\theta_{221}\left(\tfrac{\beta}{1+\alpha+\beta}\right)e_3\right\}=\\
\left(\theta_{111}+\theta_{112}+\theta_{121}+\theta_{122}\right)e_1+\left(\theta_{111}+\theta_{112}+\theta_{212}+\theta_{211}\right)e_2+\\
\left(\theta_{111}+\theta_{121}+\theta_{211}+\theta_{211}\right)e_3
\end{gathered} \tag{F.9}$$

and

$$\begin{gathered}
f\left(1,1\right)\left\{\theta_{222}\left(1+\alpha+\beta\right)+\theta_{221}\left(1+\alpha\right)+\theta_{212}\left(1+\beta\right)+\right.\\
\left.\theta_{211}+\theta_{121}\alpha+\theta_{122}\left(\alpha+\beta\right)+\theta_{112}\beta\right\}-\\
\left(1-R_{12}\right)f\left(1,1\right)\left\{\theta_{121}\alpha+\theta_{122}\left(\alpha+\beta\right)+\theta_{112}\beta\right\}-\\
\left(1-R_{12}\right)\left\{\theta_{221}\left(\tfrac{1+\alpha}{1+\alpha+\beta}\right)e_3+\theta_{212}\left(\tfrac{1+\beta}{1+\alpha+\beta}\right)e_2+\theta_{211}\left(\tfrac{e_2+e_3}{1+\alpha+\beta}\right)-\right.\\
\left.\theta_{121}\left(\tfrac{\alpha}{1+\alpha+\beta}\right)e_2-\theta_{122}\left(\tfrac{\alpha+\beta}{1+\alpha+\beta}\right)\left(e_2+e_3\right)-\theta_{112}\left(\tfrac{\beta}{1+\alpha+\beta}\right)e_3\right\}=\\
\left(\theta_{222}+\theta_{221}+\theta_{212}+\theta_{211}\right)e_1+\left(\theta_{222}+\theta_{221}+\theta_{121}+\theta_{122}\right)e_2+\\
\left(\theta_{222}+\theta_{212}+\theta_{122}+\theta_{112}\right)e_3.
\end{gathered} \tag{F.10}$$

Next, rewrite equation (F.9) as

$$\left(\theta_{221}-\theta_{112}\right)\left[\left(\frac{\beta}{1+\alpha+\beta}\right)\left(e_1+e_2+e_3\right)-e_3\right]+$$

$$\begin{gathered}(\theta_{212}-\theta_{121})\left[\left(\frac{\alpha}{1+\alpha+\beta}\right)(e_1+e_2+e_3)-e_2\right]- \\ (\theta_{211}-\theta_{122})\left[\left(\frac{1}{1+\alpha+\beta}\right)(e_1+e_2+e_3)-e_1\right]+ \\ \left(\frac{1-R_{12}}{R_{12}}\right)f(1,1)\{\theta_{212}\alpha+\theta_{211}(\alpha+\beta)+\theta_{221}\beta\}+ \\ \left(\frac{1-R_{12}}{R_{12}}\right)\{\theta_{112}\left(\frac{1+\alpha}{1+\alpha+\beta}\right)e_3+\theta_{121}\left(\frac{1+\beta}{1+\alpha+\beta}\right)e_2+ \\ \theta_{122}\left(\frac{e_2+e_3}{1+\alpha+\beta}\right)-\theta_{212}\left(\frac{\alpha}{1+\alpha+\beta}\right)e_2- \\ \theta_{211}\left(\frac{\alpha+\beta}{1+\alpha+\beta}\right)(e_2+e_3)-\theta_{221}\left(\frac{\beta}{1+\alpha+\beta}\right)e_3\}=0\end{gathered} \tag{F.11}$$

and rewrite equation (F.10) as

$$\begin{gathered}-(\theta_{221}-\theta_{112})\left[\left(\frac{\beta}{1+\alpha+\beta}\right)(e_1+e_2+e_3)-e_3\right]- \\ (\theta_{212}-\theta_{121})\left[\left(\frac{\alpha}{1+\alpha+\beta}\right)(e_1+e_2+e_3)-e_2\right]+ \\ (\theta_{211}-\theta_{122})\left[\left(\frac{1}{1+\alpha+\beta}\right)(e_1+e_2+e_3)-e_1\right]+ \\ (R_{12}-1)f(1,1)\{\theta_{121}\alpha+\theta_{122}(\alpha+\beta)+\theta_{112}\beta\}+ \\ (R_{12}-1)\{\theta_{221}\left(\frac{1+\alpha}{1+\alpha+\beta}\right)e_3+\theta_{212}\left(\frac{1+\beta}{1+\alpha+\beta}\right)e_2+ \\ \theta_{211}\left(\frac{e_2+e_3}{1+\alpha+\beta}\right)-\theta_{121}\left(\frac{\alpha}{1+\alpha+\beta}\right)e_2- \\ \theta_{122}\left(\frac{\alpha+\beta}{1+\alpha+\beta}\right)(e_2+e_3)-\theta_{112}\left(\frac{\beta}{1+\alpha+\beta}\right)e_3\}=0.\end{gathered} \tag{F.12}$$

Using $f(1,1)=\frac{e_1+e_2+e_3}{1+\alpha+\beta}$ in equations (F.11) and (F.12), it is easy to verify that these two conditions can hold simultaneously iff R_{12} satisfies (66). Solving (F.12) for R_{12} also yields (66). Thus we have the desired steady state.

Part (b). This follows directly from (66).

APPENDIX G
Proof of Claims 5.1 and 5.2

Consider a "monetary" steady state, so that $R_{11} = R_{22} = 1$, and $R_{21} = 1/R_{12}$. Then excess saving in location 1 is

$$s_1(R_{12}) \equiv \theta_{111}\left[e_1 + e_2 - (1+\alpha) f(1,1)\right] + \tag{G.1}$$

$$\theta_{112}\left[e_1 + e_2 - (1+\alpha) f(1,R_{12})\right] +$$

$$\theta_{221}\left[e_1 R_{21} + e_2 - (1+\alpha) R_{21} f(R_{21},1)\right] +$$

$$\theta_{212}\left[e_1 R_{21} + e_2 - (1+\alpha) R_{21} f(R_{21},R_{12})\right] +$$

$$\theta_{111}\left[e_1 - f(1,1)\right] + \theta_{112}\left[e_1 - f(1,R_{12})\right] +$$

$$\theta_{121}\left[e_1 - f(R_{12},R_{21})\right] + \theta_{122}\left[e_1 - f(R_{12},1)\right] \equiv$$

$$(\theta_{111} + \theta_{112} + \theta_{211} + \theta_{212})\left[\left(\frac{\beta}{1+\alpha+\beta}\right)(e_1+e_2+e_3) - e_3\right] -$$

$$(\theta_{111} + \theta_{112} + \theta_{121} + \theta_{122})\left[\left(\frac{1}{1+\alpha+\beta}\right)(e_1+e_2+e_3) - e_1\right] -$$

$$\left(\frac{1-R_{12}}{R_{12}}\right)\{\theta_{112}\left(\frac{1+\alpha}{1+\alpha+\beta}\right)e_3 - \theta_{211}\left(\frac{\beta}{1+\alpha+\beta}\right)e_1 -$$

$$\theta_{212}\left(\frac{\beta}{1+\alpha+\beta}\right)e_1 + \theta_{212}\left(\frac{1+\alpha}{1+\alpha+\beta}\right)e_3 +$$

$$\theta_{112}\left(\frac{1}{1+\alpha+\beta}\right)e_3 + \theta_{121}\left(\frac{1}{1+\alpha+\beta}\right)e_2 +$$

$$\theta_{122}\left(\frac{1}{1+\alpha+\beta}\right)(e_2+e_3)\},$$

where the second equality follows from Lemma F.0.1. Similarly, excess saving in location 2 is

$$s_2(R_{12}) \equiv \theta_{222}\left[e_1 + e_2 - (1+\alpha) f(1,1)\right] + \tag{G.2}$$

$$\theta_{221}\left[e_1 + e_2 - (1+\alpha) f(1,R_{21})\right] +$$

$$\theta_{122}\left[e_1R_{12}+e_2-(1+\alpha)R_{12}f(R_{12},1)\right]+$$

$$\theta_{121}\left[e_1R_{12}+e_2-(1+\alpha)R_{12}f(R_{12},R_{21})\right]+$$

$$\theta_{222}\left[e_1-f(1,1)\right]+\theta_{221}\left[e_1-f(1,R_{21})\right]+$$

$$\theta_{212}\left[e_1-f(R_{21},R_{12})\right]+\theta_{211}\left[e_1-f(R_{21},1)\right]\equiv$$

$$(\theta_{222}+\theta_{221}+\theta_{122}+\theta_{121})\left[\left(\frac{\beta}{1+\alpha+\beta}\right)(e_1+e_2+e_3)-e_3\right]-$$

$$(\theta_{222}+\theta_{221}+\theta_{212}+\theta_{211})\left[\left(\frac{1}{1+\alpha+\beta}\right)(e_1+e_2+e_3)-e_1\right]-$$

$$(R_{12}-1)\{\theta_{221}\left(\frac{1+\alpha}{1+\alpha+\beta}\right)e_3-\theta_{122}\left(\frac{\beta}{1+\alpha+\beta}\right)e_1-$$

$$\theta_{121}\left(\frac{\beta}{1+\alpha+\beta}\right)e_1+\theta_{121}\left(\frac{1+\alpha}{1+\alpha+\beta}\right)e_3+$$

$$\theta_{221}\left(\frac{1}{1+\alpha+\beta}\right)e_3+\theta_{212}\left(\frac{1}{1+\alpha+\beta}\right)e_2+$$

$$\theta_{211}\left(\frac{1}{1+\alpha+\beta}\right)(e_2+e_3)\},$$

where again the second equality follows from Lemma F.0.1.

Now suppose (without loss of generality) that $R_{12}<1$ holds. Then

$$s_1(R_{12})>(<)$$

$$(\theta_{111}+\theta_{112}+\theta_{211}+\theta_{212})\left[\left(\frac{\beta}{1+\alpha+\beta}\right)(e_1+e_2+e_3)-e_3\right]- \quad \text{(G.3)}$$

$$(\theta_{111}+\theta_{112}+\theta_{121}+\theta_{122})\left[\left(\frac{1}{1+\alpha+\beta}\right)(e_1+e_2+e_3)-e_1\right]$$

is satisfied iff

$$(1+\alpha)e_3(\theta_{112}+\theta_{212})-\beta e_1(\theta_{211}+\theta_{212})+ \quad \text{(G.4)}$$

$$e_2(\theta_{121}+\theta_{122})+(\theta_{112}+\theta_{122})e_3<(>)\ 0.$$

Similarly,

$$s_2(R_{12}) > (<)$$

$$(\theta_{222}+\theta_{221}+\theta_{122}+\theta_{121})\left[\left(\frac{\beta}{1+\alpha+\beta}\right)(e_1+e_2+e_3)-e_3\right]- \tag{G.5}$$

$$(\theta_{222}+\theta_{221}+\theta_{212}+\theta_{211})\left[\left(\frac{1}{1+\alpha+\beta}\right)(e_1+e_2+e_3)-e_1\right]$$

holds iff

$$(1+\alpha)\,e_3(\theta_{221}+\theta_{121})-\beta e_1(\theta_{122}+\theta_{121})+ \tag{G.6}$$

$$e_2(\theta_{212}+\theta_{211})+(\theta_{221}+\theta_{211})\,e_3 > (<)\ 0.$$

We may now note that, if $R_{12} < 1$ holds, then

$$s_1(R_{12})+s_2(R_{12}) > (<)$$

$$\left[\left(\frac{\beta}{1+\alpha+\beta}\right)(e_1+e_2+e_3)-e_3\right]- \tag{G.7}$$

$$\left[\left(\frac{1}{1+\alpha+\beta}\right)(e_1+e_2+e_3)-e_1\right]=$$

$$[e_1+e_2-(1+\alpha)\,f(1,1)]+[e_1-f(1,1)]$$

is satisfied if (G.4) and (G.6) hold (fail). Thus aggregate savings may either rise or fall as a result of spatial separation and limited communication.

To establish Claims 4.1 and 4.2, we now assume that

$$[e_1+e_2-(1+\alpha)\,f(1,1)]+[e_1-f(1,1)]=0, \tag{G.8}$$

so that the analog centralized economy is *neither* classical nor Samuelsonian.[1] In addition, we assume that $\theta_{122}=\theta_{212}$ and $\theta_{121}=\theta_{211}$. It then follows that $s_1(1)=s_2(1)=0$. Under these assumptions, $R_{12}<1$ holds if

$$(\theta_{211}-\theta_{122})\left\{\left[\left(\frac{\alpha}{1+\alpha+\beta}\right)(e_1+e_2+e_3)-e_2\right]+\right. \tag{G.9}$$

$$\left.\left[\left(\frac{1}{1+\alpha+\beta}\right)(e_1+e_2+e_3)-e_1\right]\right\}+$$

[1] That is, the analog centralized economy has a *unique* steady state with $R=1$.

$$(\theta_{112} - \theta_{221}) \left[\left(\frac{\beta}{1+\alpha+\beta} \right) (e_1 + e_2 + e_3) - e_3 \right] > 0.$$

Then it follows that $s_1(R_{12}) > (<)\ 0$ and $s_2(R_{12}) > (<)\ 0$ hold under these assumptions if (G.9) is satisfied, and if (G.4) and (G.6) both hold (fail).

Under the assumptions made thus far (G.4) holds (fails) if

$$(1+\alpha) e_3 (\theta_{112} + \theta_{122}) - (\theta_{221} + \theta_{122}) (\beta e_1 - e_2) + \tag{G.10}$$

$$(\theta_{112} + \theta_{122}) e_3 < (>)\ 0.$$

Condition (G.6) holds (fails) if

$$(1+\alpha) e_3 (\theta_{221} + \theta_{211}) - (\theta_{211} + \theta_{122}) (\beta e_1 - e_2) + \tag{G.11}$$

$$(\theta_{221} + \theta_{211}) e_3 > (<)\ 0.$$

It is straightforward to choose parameter values so that (G.9) holds, and so that (G.10) and (G.11) either both hold or both fail. Thus the decentralized economy can have either $s_1(R_{12}) > 0$ and $s_2(R_{12}) > 0$, or $s_1(R_{12}) < 0$ and $s_2(R_{12}) < 0$. This is true even though the analog centralized economy is neither classical nor Samuelsonian.

It is therefore apparent that, by small variations in parameters, we can set the term $[e_1 + e_2 - (1+\alpha) f(1,1)] + [e_1 - f(1,1)]$ either slightly positive or slightly negative without altering the previous argument. This establishes Claims 4.1 and 4.2.