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Abstract: Separating cyclical movement from trend growth at seasonal and business cycle frequencies is important to macroeconomic research. At business cycle frequencies, time trends, first differences and the more recent Hodrick-Prescott (HP) filter are used to separate trends from cycles. At seasonal frequencies, ad-hoc methods like the Census Bureau's X-11 seasonal filter are applied. This paper reviews the criteria for permanent cycles in systems characterized by difference equations and looks at the effect of filtering data which exhibit permanent cyclicality. Second order moving averages with complex unit roots at appropriate frequencies are used to filter data at seasonal and business cycle frequencies; and spectral analysis of the filtered data is used to illustrate the effect. The X-11 seasonal filter and the HP filter are also discussed in this framework. As with any filter that is applied to data where the data generating process is unknown, filtering for specific frequencies can induce cycles at harmonics of the fundamental frequency.

JEL Classification: C8, E3

Keywords: Seasonal, Filters, Cycles, Unit Roots

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Filtering Permanent Cycles with Complex Unit Roots

Separating trend macroeconomic growth from cyclical fluctuations has been a major preoccupation of economists. Whereas removing seasonal fluctuations from data generates little controversy, at the business cycle frequency this is not so. Controversy centers around whether business cycle fluctuations are driven by demand shocks or permanent shocks to technology, each with its own attendant implications for counter cyclical government policy. If business cycles are predominantly demand driven - that is, a result of reductions in aggregate demand due to “animal spirits,” then Keynesian counter cyclical stimulation may be valid policy. If fluctuations in growth rate reflect primarily random changes in productivity due to fluctuations in technological innovations, then *laissez-faire* may be the appropriate government policy. To test each of these arguments, economists attempt to isolate trend growth from cyclical fluctuations and use the filtered data to support theory or establish stylized facts. Trend removal options include estimating linear time trends, first differencing of data and the more recent Hodrick-Prescott (HP) filter. Cogley and Nason (1995) have shown that when the HP filter is used on data which does not contain business cycle frequencies, the filter can induce these cycles into the filtered data. This criticism can be extended to any filter applied when the data generating function is unknown.

This paper reviews the necessary conditions for periodic cycles in systems characterized by linear difference or differential equations and shows the impact of filtering for specific frequencies. This exposition might serve as a bridge to resolving some of the differences or controversies regarding competing methodologies for trend or cycle removal.

The existence of both complex and real unit roots can induce growth cycles similar to those observed in some economic data. Filtering some macroeconomic time series data with moving averages containing appropriate complex unit roots removes cycles at a specified frequency. Thus, where known cycles predominate in data, for example in data that have not been seasonally unadjusted, filters with complex unit roots can be designed to remove them. As is the case with all filters, some distortion can occur by the introduction of harmonic frequencies in the filtered data. Some filters such as the X-11 aim at removing both fundamental frequencies and its harmonics.

Analysis in the Complex Plane

Physical and economic systems can be modeled in the time domain as simultaneous differential equations in continuous time or difference equations in discrete time. Physical systems are governed by laws which follow known differential equations. Economic agents decision rules for utility or profit maximization can also typically be represented as difference equations in discrete time. Control engineers have found it useful and tractable to analyze the characteristics of these systems in the complex plane through a linear transformation - Laplace transform in continuous time and Z - transforms in discrete time. Analysis in the complex plane has certain advantages. One of these advantages is the representation of the system in transfer function form which allows analysis of response to different disturbances via multiplication of rational polynomials instead of complicated convolutions which are usually necessary when working in the time domain. Another advantage is that, by observing the location of the roots of the characteristic equation or the poles[1] of the transfer function in the complex plane, the transient response of the system to disturbances can be predicted.

Laplace or Z transforms of the equations produce a characteristic polynomial whose roots provide information about the systems response to disturbances, including the stability of the system. For stability, the roots of the characteristic equation must lie in the left half of the complex plane (negative real values) for a differential equation and inside the unit circle for the difference equation. When the roots of the characteristic equation are complex, the system response to disturbances is cyclical.[2] Table 1 below summarizes some of the characteristics of a second order difference and differential equations.[3] The quadratic form shown is simplified and can be part of a higher order system. If the equation is an ARMA or the right hand side has additional terms, the transfer function will have a polynomial in the numerator also. For simplicity, the examples used here assume one disturbance or input term.

Knowing the cyclical response produced by various roots allows us to "design" a system of difference equations to yield a specified cyclical response. For example, if we wanted to design a system which produces cycles with a period of 60 (or every five years if one period is considered a month), the natural frequency would be $2\pi/60$, or 0.1047 radians. We can choose α and β to give us a system with the appropriate roots. Since α must be less than Cos ω_n for stability, choose α close to 1.0, say .98, then $\beta = 0.103$ for a 60 period cycle.

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[1] A pole is the root in the denominator of the rational polynomial (transfer function) which makes it go to infinity. A zero is the root in the numerator of the transfer function which makes it zero.

[2] A necessary condition for complex roots is that the characteristic equation must be at least a quadratic (second order system) and the complex roots will come in pairs.

[3] The transforms shown in Table 1 reflect assumptions of zero initial conditions. Formal transformations include initial conditions in the characteristic equations.

Table 1 Comparison of features of continuous and discrete time transformations of second order systems

Continuous Time	*Discrete Time*
$\ddot{x}(t)+\varphi_1\dot{x}(t)+\varphi_2 x(t)=r(t)$	$y_{t+2}+\varphi_1 y_{t+1}+\varphi y_t=\varepsilon_t$
Laplace Transform	*Z transform*
$s^2X(s)+\varphi_1 sX(s)+\varphi_2 X(s)=R(s)$	$z^2Y(z)+\varphi_1 zY(z)+\varphi_2 Y(z)=\mathrm{E}(z)$
Transfer Function	
$G(s)=\dfrac{X(s)}{R(s)}=\dfrac{1}{s^2+\varphi_1 s+\varphi_2}$	$G(z)=\dfrac{Y(z)}{\mathrm{E}(z)}=\dfrac{1}{z^2+\varphi_1 z+\varphi_2}$
Complex Roots	
$G(s)=\dfrac{1}{(s+\alpha-\beta i)(s+\alpha+\beta i)}$	$G(z)=\dfrac{1}{(z+\alpha-\beta i)(z+\alpha+\beta i)}$
Stability Criterion	
$\alpha>0$	$\lvert\alpha+\beta i\rvert<1$
Natural Frequency	
$\omega_n=\beta$	$\omega_n=Cos^{-1}\left(\dfrac{\alpha}{\sqrt{\alpha^2+\beta^2}}\right)$
Period of Oscillation	
$\dfrac{2\pi}{\omega_n}$	$\dfrac{2\pi}{\omega_n}$

This gives us a difference equation of the form

$$y_{t+2}-1.96y_{t+1}+0.9854y_t=\varepsilon_t$$

$$\textit{or in more familiar form} \qquad (1)$$

$$y_t=1.96y_{t-1}-0.9854y_{t-2}+\varepsilon_t$$

The response of the above system to a disturbance is cyclical with a 60 period cycle but damped. Figure 1 shows the response to a unit impulse.

If we choose a root exactly on the unit circle, then α would be equal to Cos ω_n or 0.99452, making β equal to 0.10453. Then the difference equation would be of the form

$$y_{t+2} - 1.98904 y_{t+1} + y_t = \varepsilon_t$$

$$\text{or in more familiar form} \tag{2}$$

$$y_t = 1.98904 y_{t-1} - y_{t-2} + \varepsilon_t$$

The response of the above system to a disturbance is an undamped 60 period cycle. Figure 2 shows the impulse response of this function. As with a real unit root, the impact of the shock is permanent, but because they are complex it is also cyclical. The amplitude of the response at any period can be computed by the relationship in equation 3.

$$Dynamic\ Multiplier\ (at\ time\ t) = R^{t-1} \frac{Sin(t\theta)}{Sin(\theta)} \tag{3}$$

In this example with a period of 60, θ is 6 degrees. This bounds the multiplier at period 15 at $1/\text{Sin}\theta$ or 9.567. For smaller angles (longer periods, lower frequencies) the multiplier increases. In the limit with two real unit roots, we get a ramp response or infinite multiplier.

Figure 3 shows a unit step response, that is a shock of 1 at t=1 held constant from then on. In this case, the 60 period cycle is still evident but the amplitude of the cycle is increased and shifted above the axis. The multiplier in this case is a summation of multipliers for each shock from time t=1 to t. The initial half cycle produces “constructive interference” which increases the amplitude of the unit step response above the amplitude of the unit impulse response to about 175. After one half cycle, the response of additional unit impulses provide “destructive interference” which limits the growth of the step response.

Now consider the effect of having a cubic difference equation with two complex and one real unit roots. The equivalent Z-transform of the characteristic equation can be obtained by multiplying the original quadratic form by (z-1). The equivalent difference equation becomes

$$
\begin{gathered}
y_t = 2.98904 y_{t-1} - 2.98904 y_{t-2} + y_{t-3} + \varepsilon_t \\
\textit{which can also be written} \\
\Delta y_t = 1.98904 \Delta y_{t-1} - \Delta y_{t-2} + \Delta \varepsilon_t
\end{gathered} \tag{4}
$$

The unit impulse response of this system is exactly the same as the unit step response of the quadratic with two complex unit roots shown in Figure 3. The intuition can be discerned from the fact that the impulse response of the real unit root is a step function, so that the complex part of the system responds to a step. If we apply a unit step input to this system we get a "growth cycle" as in Figure 4 with cyclical downturns every 60 periods. Instead of a unit step, we can shock the system with a series of random shocks from a uniform distribution in the interval [0,1]. The response resembles the unit step response, except that the cycles are not as uniform in depth and peak values depend on the shocks. Figures 5 and 6 show the response for two different drawings of shocks.

So far there is no reference made to actual economic models, or whether such models do result in complex (or real) unit roots. The intent up to now is to illustrate how these models can respond in a cyclical manner to shocks. The response of the AR(3) system in equation 4 to random positive shocks does resemble some macroeconomic time series with trend and cycles - real Gross Domestic Product for example. We know however that business cycles do not appear to occur at regular intervals, even though the average occurrence in the post war period is at five year intervals.

Can we filter out the cycles?

Because we know what cycles are in the model, we can apply the proper inverse filter to it. In the cubic or AR(3) example above, we can apply a first difference to remove the unit root response, or we can apply a second order moving average or MA(2) with the same roots as the complex roots to remove the 60 period cycle. The complex filter would be of the form

$$\tilde{y}_t = y_t - 1.98904 y_{t-1} + y_{t-2} \tag{5}$$

If we apply this moving average to the step response generated by the AR(3), the filtered output is the response which would have occurred with a unit root. If we apply the unit root filter, or first difference, then the filtered output will be the response of the complex portion of the AR(3). Figure 7 shows the response and the filtered data using a step input and Figure 8 shows the same for a random inputs. Note that applying the (inverse) filter removes that part of the generating function responsible for the response. Therefore, trend growth produced by the real unit root is lower without the persistence produced by the complex portion. An adjustment in the gain of the complex filter could be made to recover a “trend” of the same magnitude as the unfiltered trend. We recognize this when we use difference filtering by treating the differenced data as growth rates rather than levels.

If we assume a cycle of period 20, which would be equivalent to 5 year cycles with quarterly data, then the equation becomes

$$y_t = 2.90211 y_{t-1} - 2.90211 y_{t-2} + y_{t-3} + \varepsilon_t$$

which can also be written (6)

$$\Delta y_t = 1.90211 \Delta y_{t-1} - \Delta y_{t-2} + \Delta\varepsilon_t$$

If we draw random shocks from a uniform distribution in the interval [0, 0.5] we get the response which looks similar to the response for the 60 period unit root when scaled down appropriately. The filtered output also looks similar as shown in Figure 9.

Filtering Actual Data

Figure 10 shows the monthly construction placed in service from January, 1970 to September 1996. Figure 11 shows the log of these values, and Figure 12 shows the data after a 60 period complex filter is applied. The filtered data appears to have a slight trend with mean reverting random fluctuations.

Filtering quarterly real Gross Domestic Product (GDP) and fixed investment with a 20 period cycle unit root filter and a real unit root filter also produces interesting results. Figures 13 and 14 show the filtered data from both these experiments. Filtering the investment data for 20 period cycles produces data that appears to be mean reverting but there is evidence of a slight trend. Filtering the GDP data with the twenty period cycle filter leaves a noticeable trend. Taking first difference (unit root filter) produces data which appears to have “cyclical” persistence. Filtering with both a unit root and a 20 period cycle complex unit root filter produces data that appears to be mean reverting or at least closer to white noise.

Adding Stable Roots

The artificially created AR(3) models assumed that only the three unit roots existed. The idea of the inverse filter is to remove that part of the model which produces the specific response. Thus we can assume an AR(4) with the fourth root being real and somewhere inside the unit circle. If we filter the model response for the three unit roots, then the filtered data will reflect the response due to the fourth root. If we add a root of 0.5 to the AR(3) model in equation 4, which is equivalent to multiplying the Z-transform equivalent polynomial by (z-0.5), we get the AR(4) equation which looks like

$$y_t = 3.98904\ y_{t-1} - 4.48356\ y_{t-2} + 2.49452 y_{t-3} - 0.5 y_{t-4} + \varepsilon_t$$

$$or$$

$$\Delta y_t = 2.98904 \Delta y_{t-1} - 2.49452 \Delta y_{t-2} + \Delta \varepsilon_t$$

The step response of this difference equation is shown in Figure 15. The response is similar to the AR(3) case because the fourth root has a decaying response to shocks. When the complex roots and the real unit root are removed, the step response of the fourth root is seen as a step of magnitude two. A stable root has a decaying impulse response and a unit step response which ramps up to a step with a multiplier which depends on the root. In this example the multiplier is 2. This doubling of the step by the stationary root doubles the amplitude of the cyclical portion of the system response to 350 compared to 175 in the AR(3), as shown by the output from the unit root filter.

If we shock this system with shocks drawn from a uniform distribution on [0,1], the response and filtered response is as shown in Figure 16. Notice that the amplitude of the unit root filtered portion does is growing over time but is less than the step response.

Seasonal cycles

We can use a complex filter to filter data that has not been seasonally adjusted and compare to see the effect. Assuming a 12 period cycle, the equivalent frequency is 2π/12 or 0.5236 radians. This gives an equivalent MA(2) filter of the form

$$\tilde{y}_t = y_t - 1.732y_{t-1} + y_{t-2}$$

Spectral analysis of data gives the dominant frequencies in a Fourier expansion of the data. This can serve as a guide to filtering. Figure 17 shows log of unadjusted construction placed in service in 1992 chain weighted dollars and the spectra of the data. The horizontal axis shows the frequency as a function of the length of the sample period. The construction data are for 382 months, so a cycle of length 12 months shows up as 382/12 or 31.8 and a 6 month cycle would be at 63.7 on the horizontal axis. Low frequency spikes are evident and two spikes at 12 month and 6 month periods. The figure shows the results of running the data through a 12 month period complex filter and then through a 6 month period complex filter. As the figure shows, the obvious seasonal spikes are removed at the appropriate frequencies but the filtered data shows new spikes at higher harmonics of the fundamental frequency. The higher frequencies introduced can be interpreted as shocks which are closer to white noise. The bottom chart combines the three spectra before and after filtering.

As with any other filter, the assumption is that the endogenous propagating mechanism or the shocks driving the system contain the components which one is trying to filter. If we filter for a specific frequency, then if neither the propagation mechanism nor the shocks contain this frequency, the filtered data will be distorted. Since all periodic series can be represented by an infinite Fourier series of harmonics, any filter of a given frequency will remove that frequency from the Fourier series representation of the data series. Thus, higher harmonics will remain. Another caution in using filters is that components which are being filtered may have unknown amplitude, and filtering at a specific amplitude will also cause distortions.

X-11 Filter

Ghysels and Perron (1993) define the X-11 seasonal filter in lag notation as

$$M_1(L) = \frac{1}{9}(L^{s}+1+L^{-s})^2$$

$$\begin{aligned} HM(L) = {} & -0.019L^{6} - 0.028L^{5} + 0.066L^{3} +).147L^{2} \\ & +0.214L + 0.240 + 0.214L^{-1} + 0.147L^{-2} \\ & +0.066L^{-3} - 0.028L^{-5} - 0.019L^{-6} \end{aligned} \tag{9}$$

and

$$M_2(L) = \frac{1}{15}(L^{s} + 1\ L^{-s})(L^{2s} + L^{s} + L^{-s} + L^{-2s})$$

where s = 12. If we convert equation 9 to Z-transform notation, the equations remain unchanged because the transformation substitutes z^{-1} for L. The procedure passes the data first through the M_1 filter, then the residual is passed through the HM filter and a final estimate of the seasonal component is obtained by applying the M_2 filter. We can determine the frequencies which these filters remove by noting the roots of these polynomials. Table 2 shows the roots of each part of the filter. In particular, the real and imaginary part of each root is shown, the modulus of the root and the effective period of the cycle which each root removes. Because of the symmetry of the equations, the roots are repeated and each complex root has its conjugate. Only the absolute values of the roots are shown.

Table 2: Roots of characteristic equations of X-11 filter.

Filter	Real	Imaginary	Modulus	Period (Natural frequency)
M_1	0.9848	0.1736	1.0	36 periods
	0.9397	0.3420	1.0	18
	0.7660	0.6428	1.0	9
	0.6428	0.7660	1.0	7.2
	0.3420	0.9397	1.0	5.1
	0.1736	0.9848	1.0	4.5
HM	0.9759	0.2182	1.0	28.6
	0.8080	0.5893	1.0	10
	0.5084	0.8611	1.0	6.1
	0.3896	0.9210	1.0	5.4
M_2	0.9945	0.1045	1.0	60 periods
	0.9781	0.2079	1.0	30
	0.9511	0.3090	1.0	20
	0.9135	0.4067	1.0	15
	0.8320	0.4803	0.9607	12
	0.8090	0.5878	1.0	10
	0.7431	0.7360	1.046	8.05
	0.6691	0.7431	1.0	7.5
	0.5878	0.8090	1.0	6.67
	0.4803	0.8320	0.9607	6
	0.4067	0.9135	1.0	5.454
	0.3090	0.9511	1.0	5
	0.2694	1.005	1.041	4.8
	.2079	0.9781	1.0	4.6

Almost all the roots are of modulus 1.0, which is to be expected if cycles are being filtered. What is surprising is that both M_1 and M_2 appear to filter both a fundamental frequency and its components. Also surprising is that a seasonal filter such as X-11 has roots which filter data at business cycle frequency (i.e. 60 month). This suggests that if X-11 is applied to data which does not contain 60 period cycles, these cycles might be induced spuriously. Given the amount of data which is seasonally adjusted using the X-11 seasonal filter, this could be a serious flaw in seasonally adjusted data.

HP Filtering

The Hodrick-Prescott (HP) filter in favor with real business cycle theorists separates high frequency portions of data from low frequency or trend. Cogley and Nason (1995) show that if the data generating process does not have business cycle frequencies in it, then the use of this filter will induce cycles at these frequencies and lead to potential misinterpretation of data. Figure 18 shows the result of applying the HP filter to the unadjusted construction placed in service data shows reasonable (by “eyeball metric”) smoothing of the data into long term trends. A spectral density plot of the unfiltered, trend and filtered data shows that the primary frequencies are in fact separated by the filter. Figure 19 compares the HP filtered trend with the unadjusted and seasonally adjusted data and compares the spectral density of the seasonally unadjusted and adjusted data. Note that the HP filtered trend is smooth compared to the seasonally adjusted data. As expected the adjusted data no longer show the 12 month and higher frequency contribution. The HP filtered data on the other hand shows all high frequency contributions remaining in the data after the removal of the low frequency trend.

Summary/Conclusions

The objective of this paper was to review the role of complex roots of modulus 1.0 in generating permanent cyclical responses to shocks. Using the appropriate complex filter, we can remove cyclical components generated at specific frequencies. In doing so, filtered data will contain higher harmonics of the filtered frequency. The X-11 seasonal filter solves this problem by filtering all the harmonics of the 12 month cycle. The HP filter operates as a smother which works by minimizing the change from period to period with a penalty factor chosen to reflect the frequency of the data. The effect of the HP filter is to remove all but the lowest frequencies which exist in the data. Work by Cogley and Nason (1995) point out that uncontrolled use of the HP filter can induce spurious cycles where none existed. This is a caution applicable to all filters. The implications of complex unit root filtering mechanisms for hypothesis testing is not immediately clear, but heightened knowledge of the inner workings of filters should be of paramount interest to macroeconomists.

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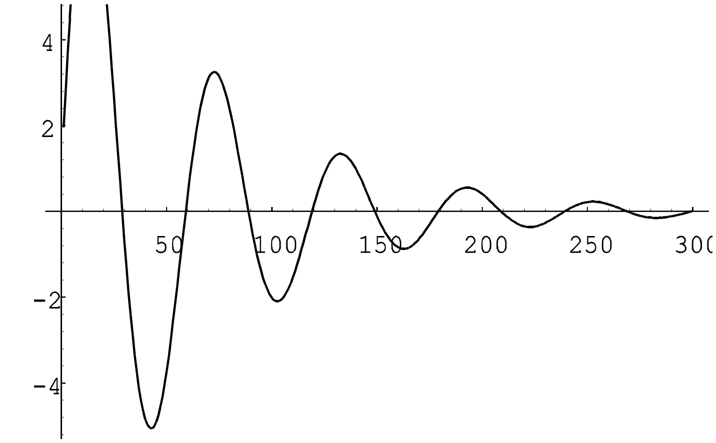


Figure 1 Impulse response of stable 60 period cycle complex unit root.

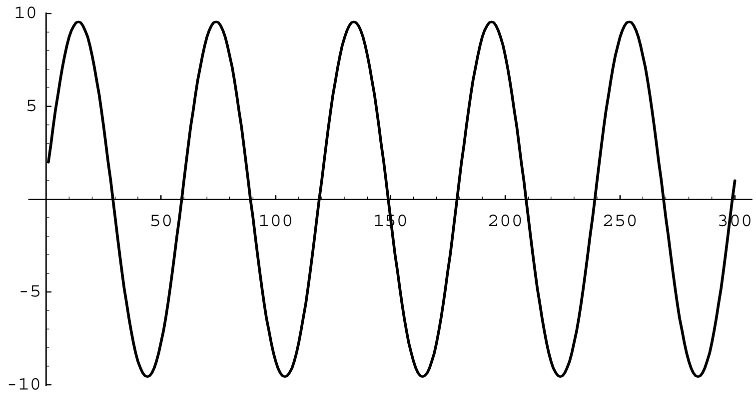


Figure 2 Impulse response of complex unit root with period of 60.

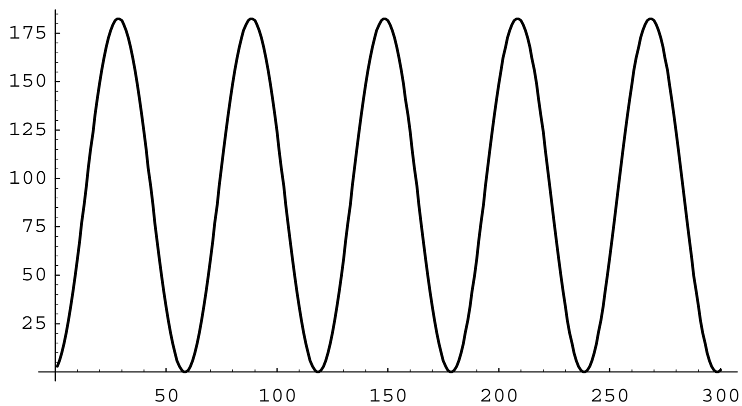


Figure 3 Step response of complex unit root

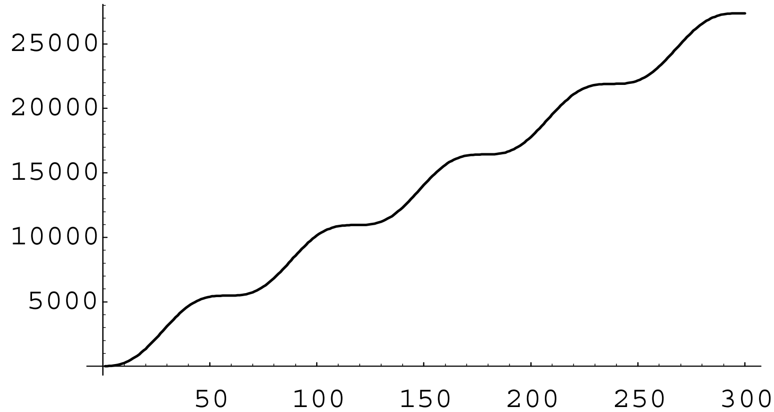


Figure 5 Step Response of AR(3) system with two complex (60 period) and one real unit root.

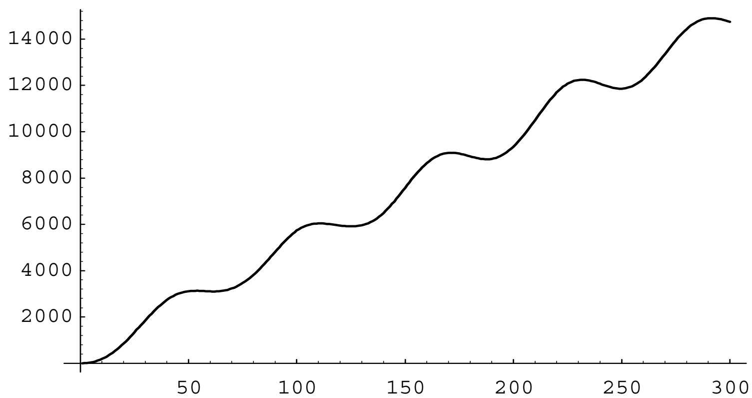


Figure 4 Response of AR(3) with two complex and one real unit roots to random shocks.

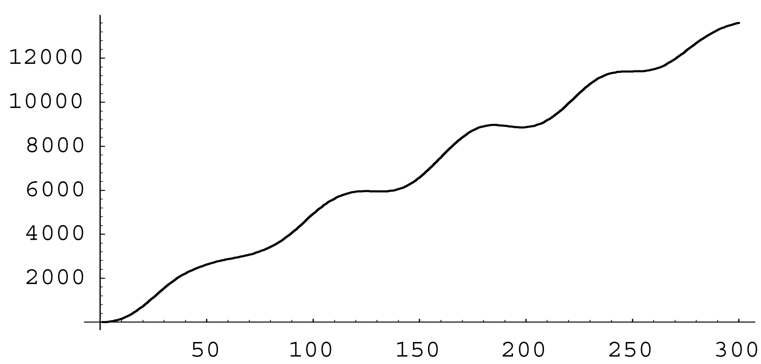


Figure 6 Response of AR(3) with two complex and one real unit root to random shocks.

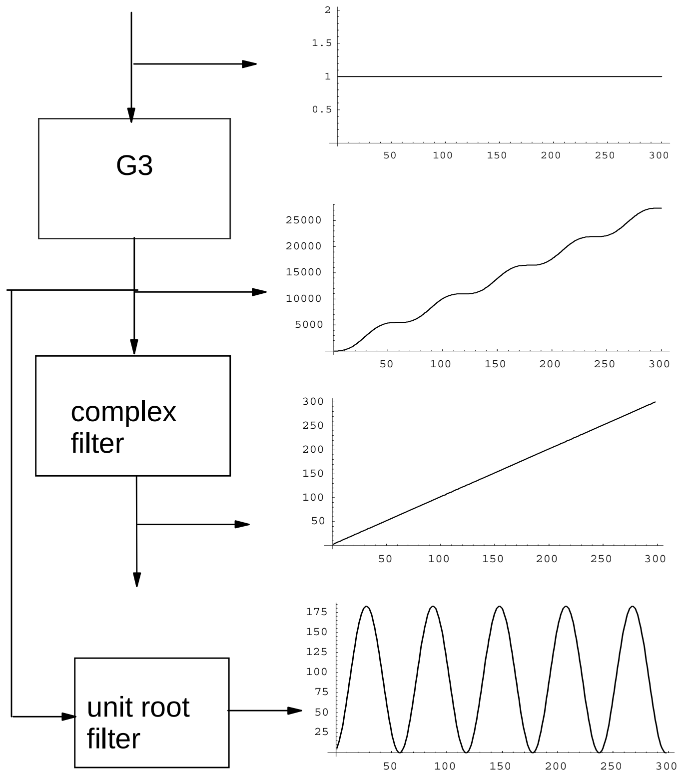


Figure 7 Output and filtered output for AR(3) with step input

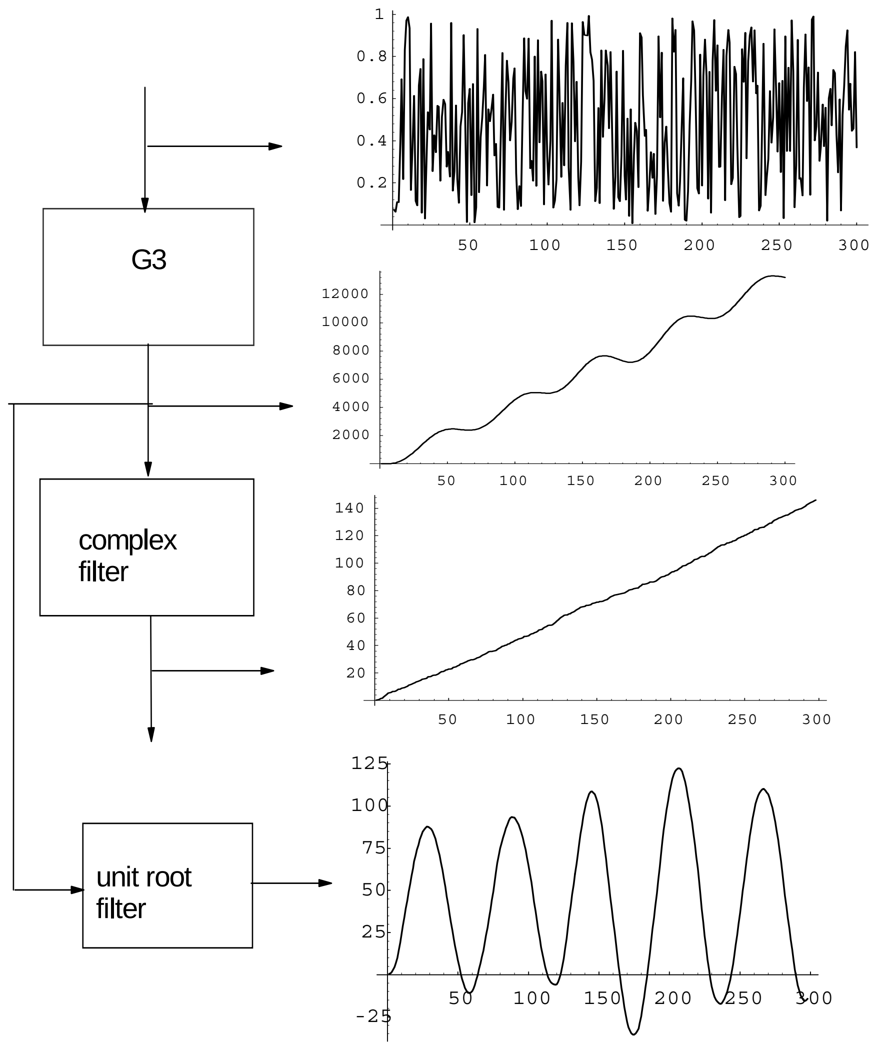


Figure 8 Output and filtered output for AR(3)

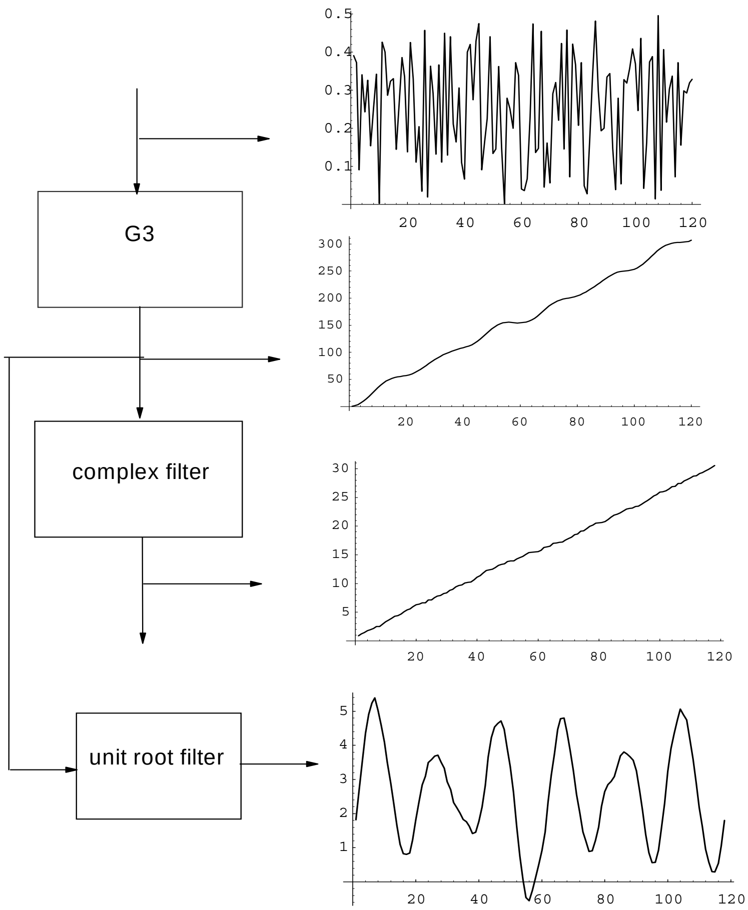


Figure 9 Response and filtered data for 20 period cycle

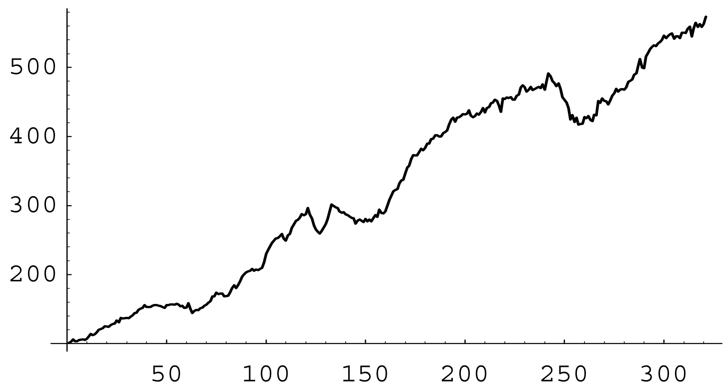


Figure 10 Construction placed in service

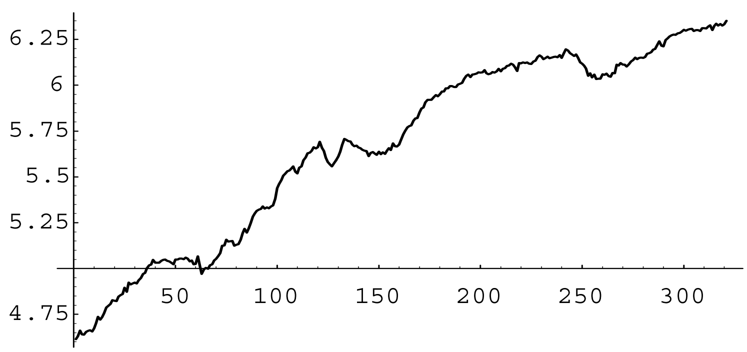


Figure 11 Log of Construction placed in service 1970-1996

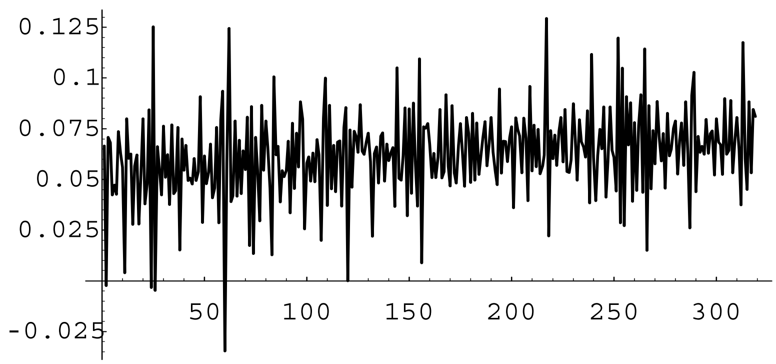


Figure 12 Construction placed in service data filtered using 60 period filter

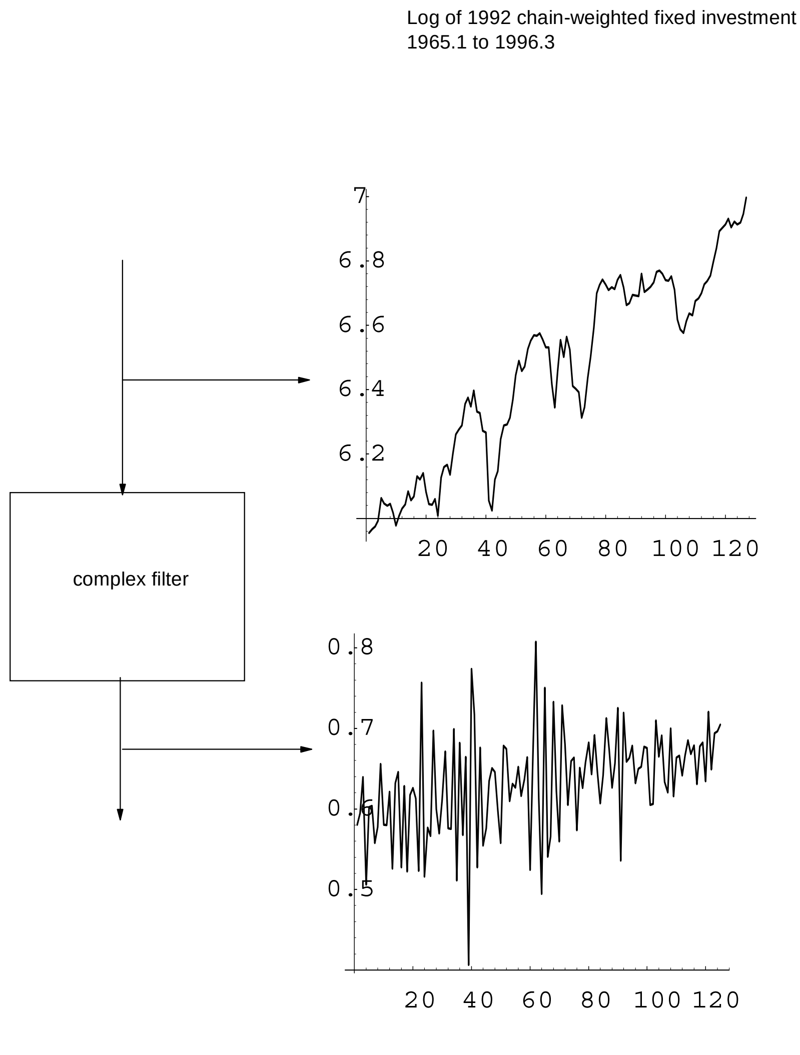


Figure 13 20 period complex cycle filter applied to quarterly fixed investment

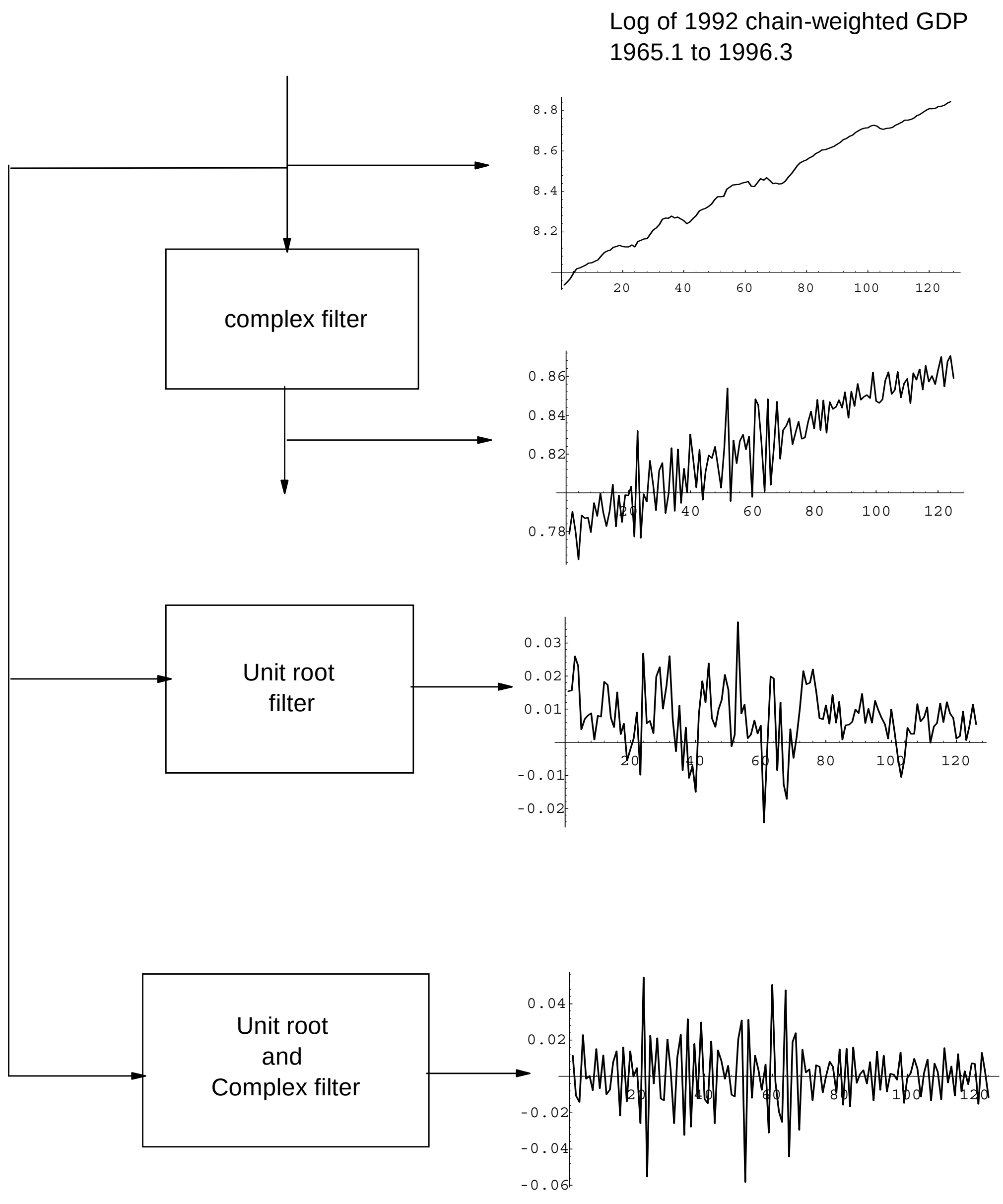


Figure 14 Unit root and 20 period cycle filters of quarterly GDP

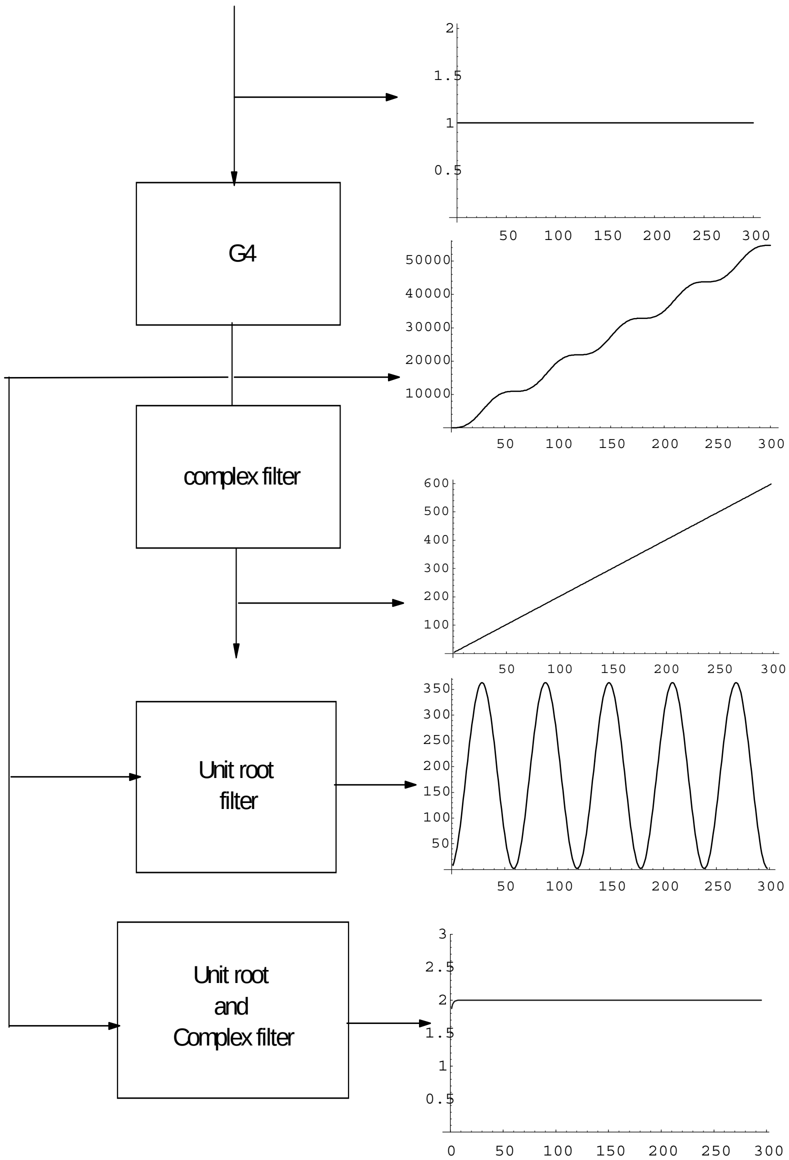


Figure 15 Fourth order difference equation or AR(4) with one stationary root.

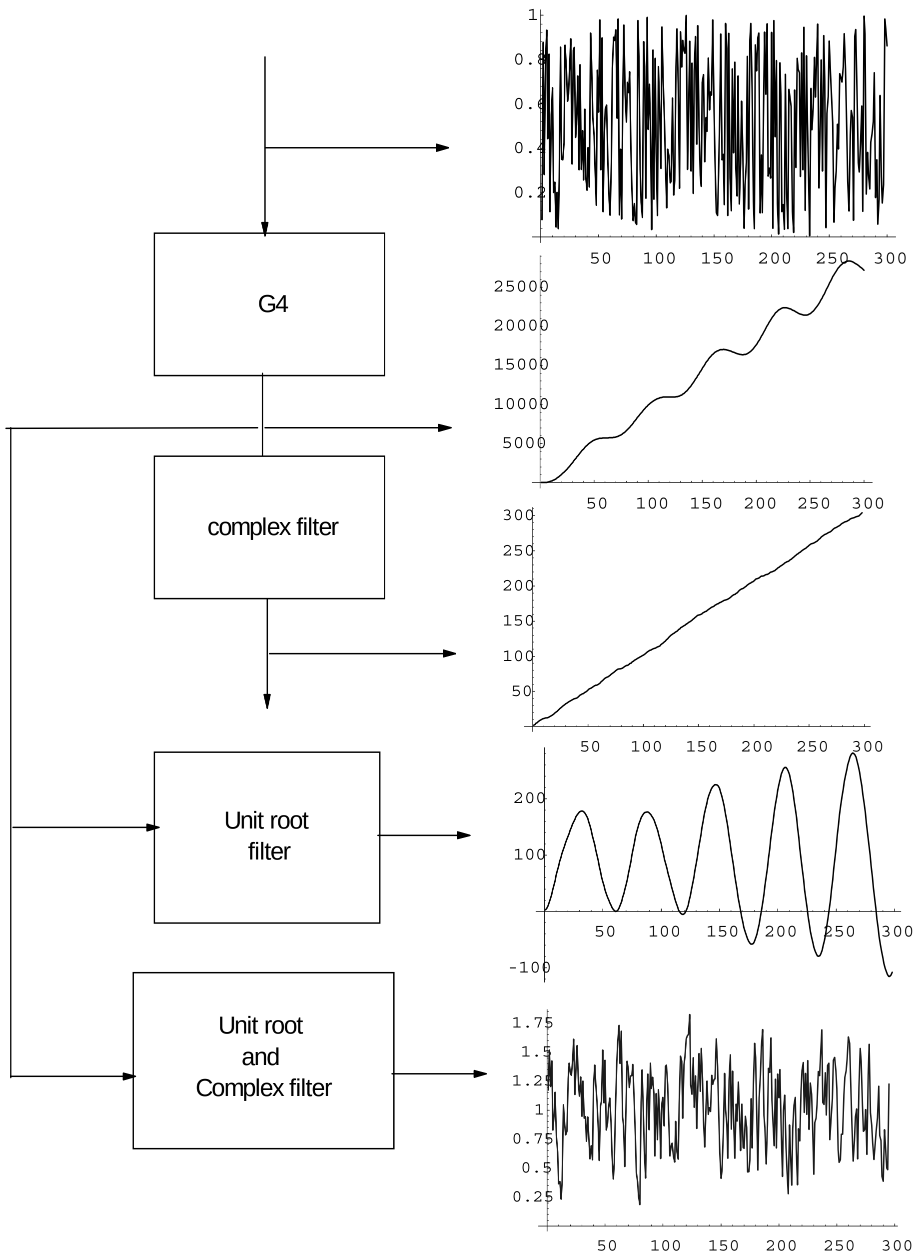


Figure 16 AR(4) Response to random shocks: complex and unit root filters

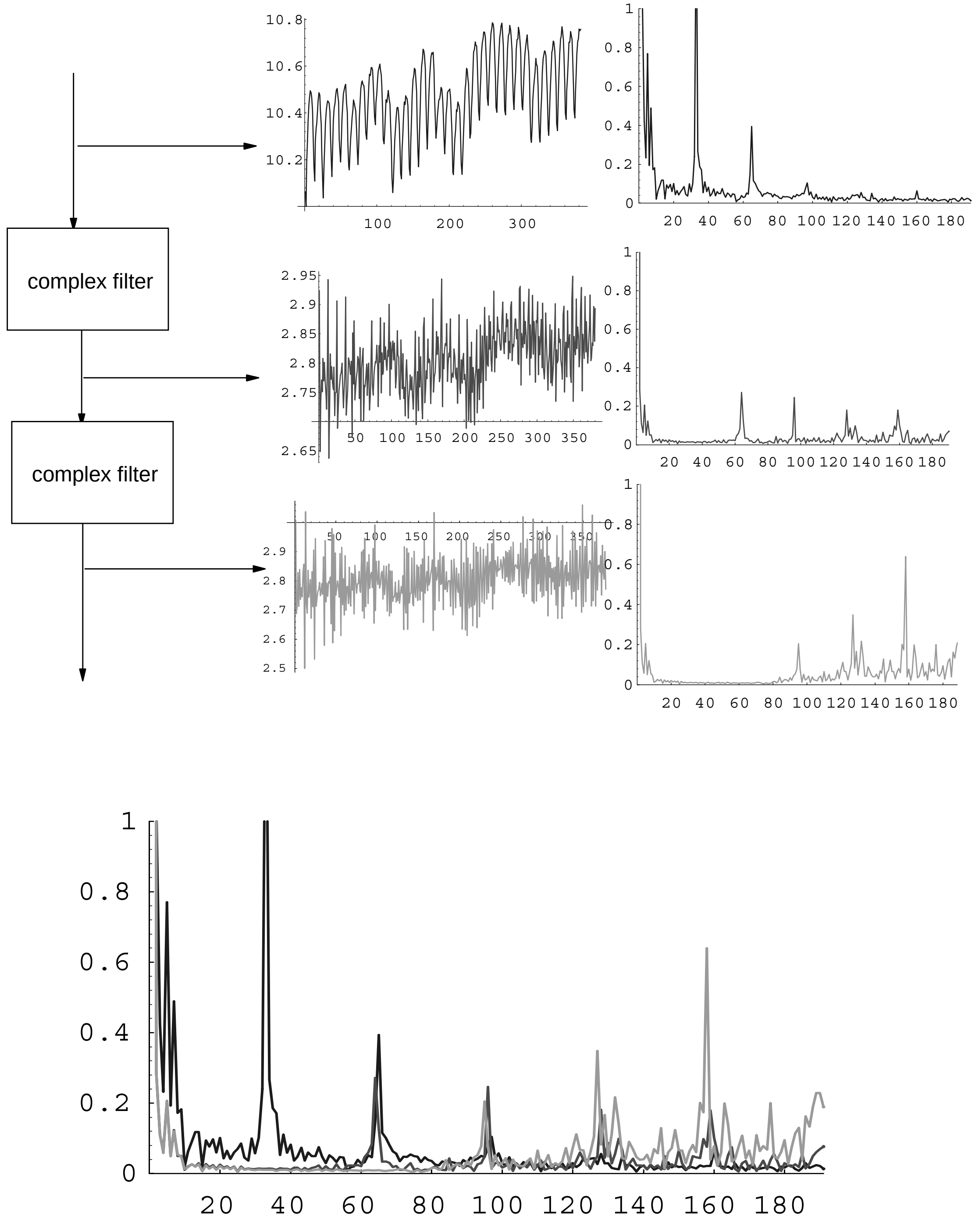


Figure 17 Filtering seasonal frequencies from construction data with complex unit root filters

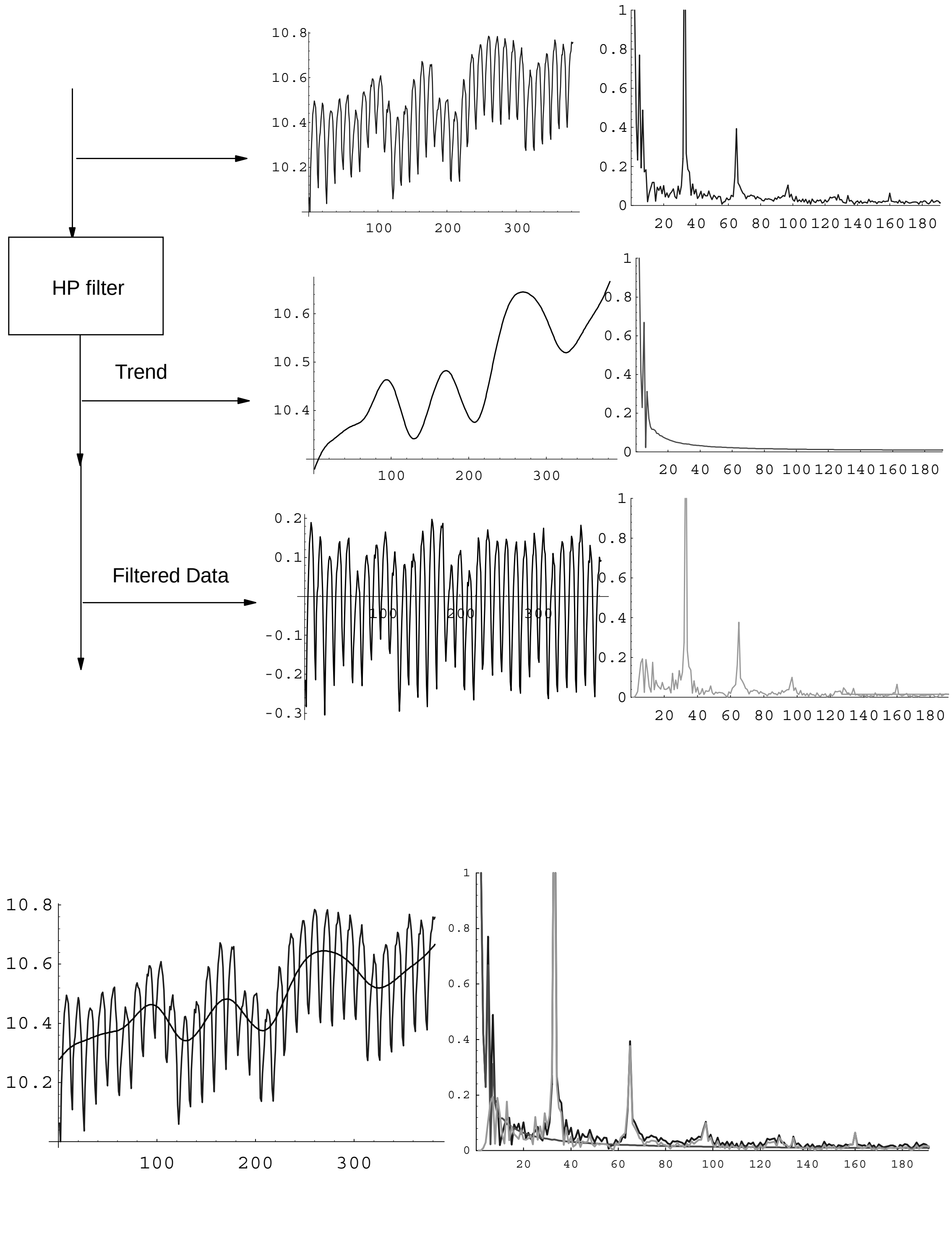
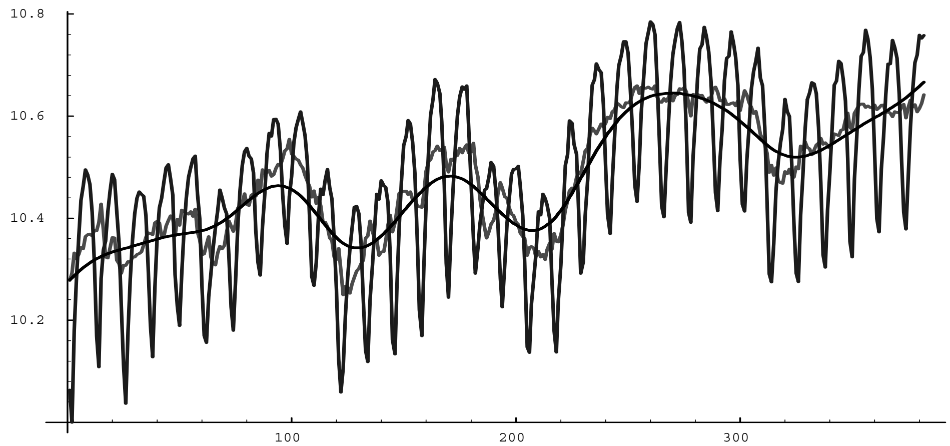


Figure 18 HP filtering of construction placed in service (unadjusted for seasonal variations)



HP filtered, seasonally adjusted (red) and unadjusted (blue) monthly construction placed in service.

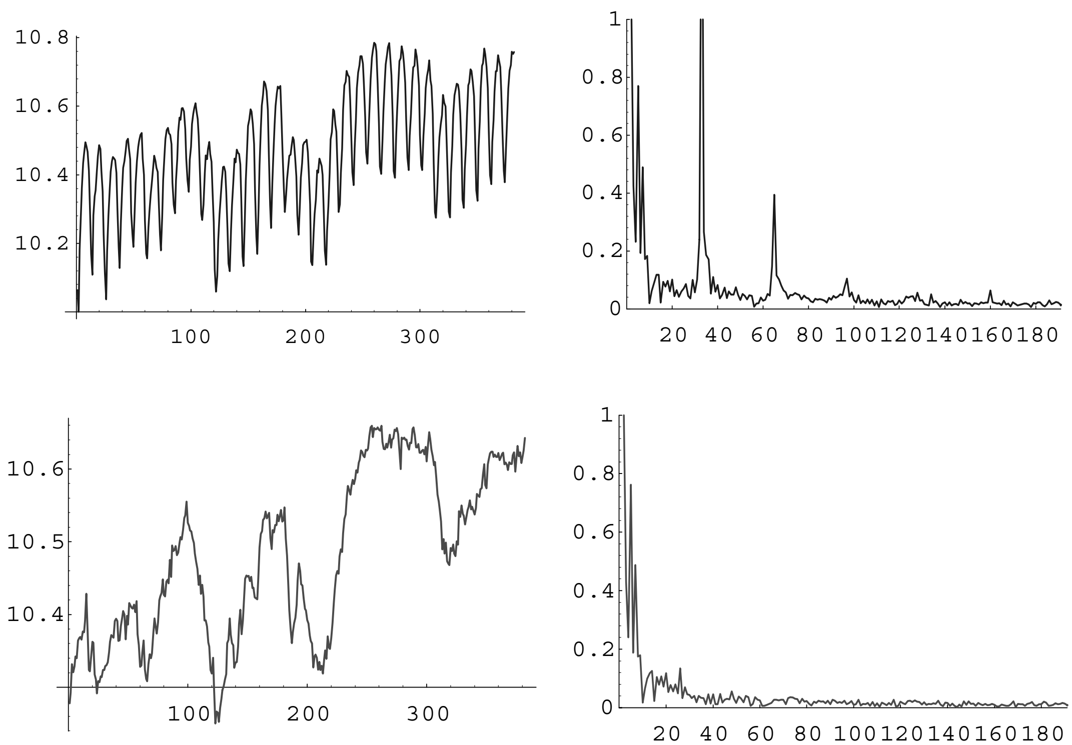


Figure 19 Comparison of monthly construction placed in service seasonally adjusted and unadjusted.